

DM559
Linear and Integer Programming

Lecture 1
Introduction

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1. Course Organization

2. Preliminaries

1. Course Organization

2. Preliminaries

Learn about:

- both the theory and the practice of Linear Algebra
- one of the most important applications of Linear Algebra:
 - Mathematical optimization: linear programming
 - Discrete optimization: integer programming

↪ Practical experience with computer software

↪ The solution to really many real-life problems will be within short reach

Contents of the Course (1/2)

(see Syllabus)

Linear Algebra: manipulation of matrices and vectors with some theoretical background

Linear Algebra

- 1 Matrices and vectors - Matrix algebra, Geometric insight
- 2 Systems of Linear Equations - Gaussian elimination
- 3 Matrix inversion and determinants
- 4 Vector spaces
- 5 Linear Transformations - Matrix representation
- 6 Diagonalization - Eigenvalues and Eigenvectors
- (7 Orthogonality or Numerical Methods)

Contents of the Course (2/2)

(see Syllabus)

Linear Programming

- 1 Introduction - Linear Programming, Notation
- 2 Linear Programming, Simplex Method
- 3 Exception Handling
- 4 Duality Theory
- 5 Sensitivity
- 6 Revised Simplex Method

Integer Linear Programming

- 7 Modeling Examples, Good Formulations, Relaxations
- 8 Well Solved Problems
- 9 Network Optimization Models (Max Flow, Min cost flow, matching)
- 10 Cutting Planes & Branch and Bound
- 11 More on Modeling

Teacher: Marco Chiarandini (<http://www.imada.sdu.dk/~marco>)

Instructor: Jens Østergaard

Schedule:

- Introductory classes: 44 hours (22 classes)
- Training classes: 50 hours
 - Exercises: 42 hours
 - Laboratory: 8 hours

Alternative views of the schedule:

- <http://mitsdu.sdu.dk/skema>, SDU Mobile
- Official course description (læserplaner)
- <http://www.imada.sdu.dk/~marco/DM559>

- BlackBoard (BB) ⇔ Main Web Page (WWW)
(link <http://www.imada.sdu.dk/~marco/DM559>)
- **Announcements** in BlackBoard
- **Discussion Board** in (BB) - allowed anonymous posting and rating
- Write to Marco (marco@imada.sdu.dk) and to instructor
- Ask peers
- You are welcome to visit me in my office in working hours (8-16)

↪ It is good to ask questions!!

↪ Let me know when you think we should do things differently! Things can be changed.

Linear Algebra Part:

AR Howard Anton and Chris Rorres. Elementary Linear Algebra. 11th Edition. 2014. Wiley

Other books:

AH Martin Anthony and Michele Harvey, Linear Algebra, Concepts and Methods. 2012. Cambridge

Le Steven J. Leon, Linear Algebra with Applications, 8th edition, Prentice Hall (2010).

FSV Computing with Python: An introduction to Python for science and engineering Claus Führer, Jan Erik Solem, Olivier Verdier

Linear and Integer Programming Part:

LN Lecture Notes

MG J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

Wo L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

Other books:

HL Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

- ...

Main Web Page (WWW) is the main reference for list of contents (ie¹, syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- software

¹ie = id est, eg = exempli gratia, wrt = with respect to

- 7.5 ECTS
- (Four) Obligatory Assignments, pass/fail, evaluation by teacher
both theoretical and practical (programming) exercises
- 4 hour written exam, 7-grade scale, external censor
(theory part)
similar to exercises in class and past exams
in June
- (language: Danish and/or English)

Obligatory Assignments

- Small practical tasks must be passed to attend the written exam
- Individual work
- They require the use of Python + a MILP Solver (2nd part) See Tools from Public Web Page

Training Sessions

- Prepare them in advance to get out the most
- Best carried out in small groups
- Exam rehearsal (in June?)

Some Numbers

DM559

Obligatory Assignments

In Blackboard	60
Registered	43
Submitted	24
Passed	19

Final exam:

Registered	19
Showed up	14
Passed	6

Reexam:

Registered	12
Showed up	6
Passed	4

DM545

Obligatory Assignments

In Blackboard	60
Registered	27
Submitted	19
Passed	18

Final exam:

Registered	29
Showed up	21
Passed	16

Reexam:

Registered	12
Showed up	4
Passed	4

Who is here?

65 registered in BlackBoard...

- Computer Science
(2nd year, 4th semester)
- Applied Mathematics?
- Math-economy?
- Returning CS

Prerequisites

- Goodwill
- Algorithms and Data Structures,
Programming

Coding small examples can help you to **understand math!**

Beside:

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand

you can learn **by doing** by **interacting with Python**.

- jupyter.org/ipython (= interactive python)
- Python 2.7 or 3.5?

1. Course Organization

2. Preliminaries

- Notation
- Matrices and vectors:
 - matrix arithmetic operations (addition, subtraction, and multiplication)
 - scalar multiplication and transposition.

- A **set** is a collection of objects. eg.: $A = \{1, 2, 3\}$
- $A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}$
('|' reads 'such that')
- $B = \{x \mid x \text{ is a reader of this book}\}$
- $x \in A$
 x belongs to A
- set of no members: **empty set**, denoted \emptyset
- if a set S is a (**proper**) **subset** of a set T , we write $(S \subset T) \quad T \supseteq S$
 $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 30\}$
- for two sets A and B , the **union** $A \cup B$ is $\{x \mid x \in A \text{ or } x \in B\}$
- for two sets A and B , the **intersection** $A \cap B$ is $\{x \mid x \in A \text{ and } x \in B\}$
 $\{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cap B = \{2, 5\}$

Numbers

- set of real numbers: \mathbb{R}
- set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ (positive integers)
- set of all integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- set of rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
- set of complex numbers: \mathbb{C}
- absolute value (non-negative):

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

$$|a + b| \leq |a| + |b|, \quad a, b \in \mathbb{R}$$

- the set \mathbb{R}^2 is the set of ordered pairs (x, y) of real numbers (eg, coordinates of a point wrt a pair of axes, the [Cartesian plane](#))

Matrices and Vectors

- A **matrix** is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- An $n \times 1$ matrix is a **column vector**, or simply a vector:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- the set \mathbb{R}^n is the set of vectors $[x_1, x_2, \dots, x_n]^T$ of real numbers (eg, coordinates of a point wrt an n -dimensional space, the **Euclidean Space**)

Elementary Algebra: the study of symbols and the rules for manipulating symbols. It differs from **arithmetic** in the use of abstractions, such as using letters to stand for numbers that are either unknown or allowed to take on many values

- collecting up terms: eg. $2a + 3b - a + 5b = a + 8b$
- multiplication of variables: eg:

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms: eg:

$$\begin{aligned}-(a - 2b) &= -a + 2b, \\(2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\&= 2x^2 + 5xy - 12y^2\end{aligned}$$

- $a^r a^s = a^{r+s}$, $(a^r)^s = a^{rs}$, $a^{-n} = 1/a^n$,
 $a^{1/n} = x \iff x^n = a$, $a^{m/n} = (a^{1/n})^m$

- In Mathematics and Statistics, a **variable** is an alphabetic character representing a **value**, which is unknown. They are used in **symbolic** calculations.
Commonly given one-character names.
- in contrast, a **constant** or **given** or **scalar** is a known real number
- In **Computer Science**, a **variable** is a storage location paired with an associated identifier, which contains a value, which may be known or unknown.
Commonly given long, explanatory names.

- a **function** f on a set \mathcal{X} into a set \mathcal{Y} is a rule that assigns a **unique** element $f(x)$ in \mathcal{Y} to each element x in \mathcal{X} .

$$y = f(x)$$

y dependent
variable

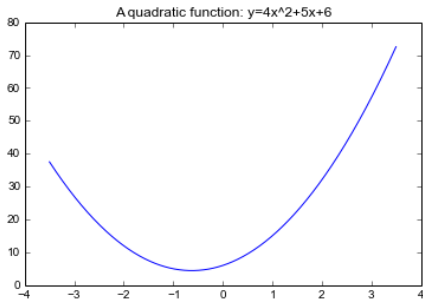
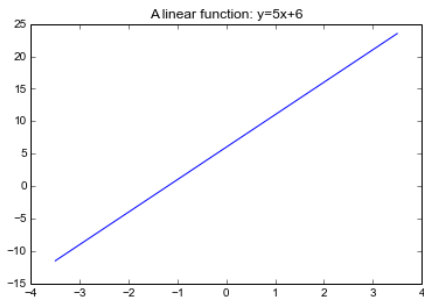
x independent
variable

- a **linear function** has only sums and scalar multiplications, that is, for variable $x \in \mathbb{R}$ and scalars $a, b \in \mathbb{R}$:

$$f(x) := ax + b$$

Graphs of Functions

The graph of a function f consists of those points in the Cartesian plane whose coordinates (x, y) are pairs of input-output values for f .



- for a linear equation: $ax + b = 0$ with $a, b \in \mathbb{R}$
the **solution** is a real number x for which the equation is true

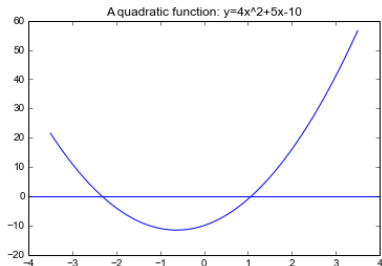
- Quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

- **closed form** or **analytical** solution:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$



- Quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

- quadratic formula:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

the term $b^2 - 4ac$ is called **discriminant**

- Solutions from discriminant:

- if $b^2 - 4ac > 0 \implies$ **two** real solutions
- if $b^2 - 4ac = 0 \implies$ **exactly one** solution: $x = -b/(2a)$
- if $b^2 - 4ac < 0 \implies$ **no real** solution but complex solutions

- Can be solved also by factorization, eg:

$$x^2 - 6x + 5 = (x - 1)(x - 5) = 0$$

then either $x - 1 = 0$ or $x - 5 = 0$.

or by completing the square $\implies a(x + d)^2 + e = 0$, eg:

i) $x^2 + 6x + 9 = 0$, and ii) $x^2 - 2x + 3 = 0$

Polynomial Equations

- A polynomial of degree n in x is an expression of the form:

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where the a_i are real constants, $a_n \neq 0$, and x is a real variable.

- $P_n(x) = 0$ has at most n solutions, eg:

$$x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3) = 0,$$

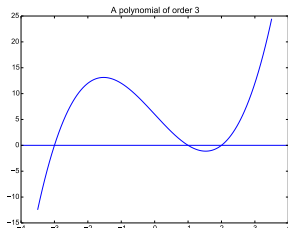
which are called roots or zeros of $P_n(x)$

- No general (closed) formula
- If α is a solution then $(x - \alpha)$ must be a factor for $P_n(x)$
We find α by trial and error and then set $(x - \alpha)Q(x)$ where $Q(x)$ is a polynomial of degree $n - 1$
- Eg, $x^3 - 7x + 6$

```
import matplotlib.pyplot as plt
import numpy as np
a=[1,0,-7,6]
P=np.poly1d(a)
print(P)
```

```
| 3
| 1 x - 7 x + 6
```

```
x = np.linspace(-3.5, 3.5, 500)
plt.plot(x, P(x), '-')
plt.axhline(y=0)
plt.title('A polynomial of order 3');
```



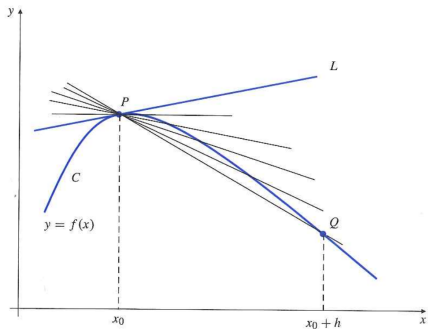
Differentiation

A line L through a point $(x_0, f(x_0))$ of f can be described by:

$$y = m(x - x_0) + f(x_0)$$

The **derivative** is the slope of the line that is tangent to the curve:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

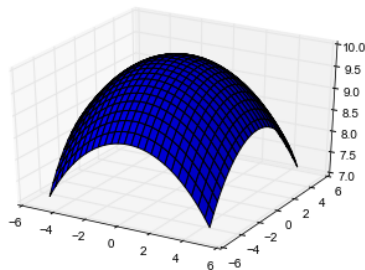


- A function f of n real variables is a rule that assigns a **unique** real number $f(x_1, x_2, \dots, x_n)$ to each point (x_1, x_2, \dots, x_n)

Example in \mathbb{R}^2 :

$$f(x, y) = \sqrt{10^2 - x^2 - y^2}$$

$$x^2 + y^2 + z^2 = 10$$



Partial Derivatives

- The **first partial derivative** of the function $f(x, y)$ with respect to the variables x and y are:

$$f_1(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h} = \frac{\partial}{\partial x} f(x, y)$$

$$f_2(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y + k) - f(x, y)}{k} = \frac{\partial}{\partial y} f(x, y)$$

- Their value in a point (x_0, y_0) is given by:

$$f_1(x_0, y_0) = \left(\frac{\partial}{\partial x} f(x, y) \right) \Big|_{(x_0, y_0)}$$

$$f_2(x_0, y_0) = \left(\frac{\partial}{\partial y} f(x, y) \right) \Big|_{(x_0, y_0)}$$

- sine and cosine functions, $\sin \theta$ and $\cos \theta$, geometrical meaning
- angles measured in **radians** rather than **degrees** ($\pi = 3.141\dots$, $\pi = 180$)
- $\cos x = \sin(x + \pi/2)$
- $\sin^2 \theta + \cos^2 \theta = 1$
- $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$
- $\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$