DM559

Linear and Integer Programming

Lecture 6
Rank and Range
Vector Spaces

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Outline

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1. Rank

2. Range

3. Vector Spaces

Rank

- Synthesis of what we have seen so far under the light of two new concepts: rank and range of a matrix
- We saw that:
  every matrix is row-equivalent to a matrix in reduced row echelon form.

Definition (Rank of Matrix)

The rank of a matrix A, $\text{rank}(A)$, is

- the number of non-zero rows, or equivalently
- the number of leading ones

in a row echelon matrix obtained from A by elementary row operations.

$\rightsquigarrow$ For an $m \times n$ matrix A,

$$\text{rank}\, A \leq \min\{m, n\},$$

where $\min\{m, n\}$ denotes the smaller of the two integers m and n.

Example

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 6 \end{bmatrix} \xrightarrow[]{\substack{R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \end{bmatrix} \xrightarrow[]{\substack{R_2' = -R_2 \\ R_3' = R_3 - R_2}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightsquigarrow \operatorname{rank}(M) = 2$

Extension of the main theorem

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. A *is invertible*
2. $A\mathbf{x} = \mathbf{b}$ *has a unique solution for any* $\mathbf{b} \in \mathbb{R}$
3. $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution,* $\mathbf{x} = \mathbf{0}$
4. *the reduced row echelon form of* A *is* I.
5. $|A| \neq 0$
6. *the rank of* A *is* n

Rank and Systems of Linear Equations

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 1 \\ 2x & + & 3y & & & = & 5 \\ 3x & + & 5y & + & z & = & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 4 \end{array}\right] \xrightarrow[R_3'=R_3-3R_1]{R_2'=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow[R_3'=R_3-R_2]{R_2'=-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 1 \\ & & x & + & 2z & = & -3 \\ 0x & + & 0y & + & 0z & = & -2 \end{array}$$

It is inconsistent!

The last row is of the type $0 = a, a \neq 0$, that is, the augmenting matrix has a leading one in the last column

$\text{rank}(A) = 2 \neq \text{rank}(A \mid \mathbf{b}) = 3$

1. A system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of the augmented matrix is precisely the same as the rank of the matrix A.

2. If an $m \times n$ matrix A has rank m, the system of linear equations, $A\mathbf{x} = \mathbf{b}$, will be consistent for all $\mathbf{b} \in \mathbb{R}^n$

- Since A has rank m then there is a leading one in every row. Hence $[A \mid \mathbf{b}]$ cannot have a row $[0, 0, \ldots, 0, 1] \implies \operatorname{rank} A \not< \operatorname{rank}(A \mid \mathbf{b})$
- $[A \mid \mathbf{b}]$ has also m rows $\implies \operatorname{rank}(A) \not> \operatorname{rank}(A \mid \mathbf{b})$
- Hence, $\operatorname{rank}(A) = \operatorname{rank}(A \mid \mathbf{b})$

Example

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \operatorname{rank}(B) = 3$$

Any system $B\mathbf{x} = \mathbf{d}$ in 4 unknowns and 3 equalities with $\mathbf{d} \in \mathbb{R}^3$ is consistent.

Since $\operatorname{rank}(A)$ is smaller than the number of variables, then there is a non-leading variable. Hence infinitely many solutions!

Example

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 3 & 0 & 4 & 0 & -28 \\ 0 & 0 & 1 & 2 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}([A|\mathbf{b}]) = 3 < 5 = n$

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 3x_2 & & & + & 4x_4 & & & = & -28 \\ & & & & x_3 & + & 2x_4 & & & = & -14 \\ & & & & & & & & x_5 & = & 5 \end{array}$$

x_1, x_3, x_5 are leading variables; x_2, x_4 are non-leading variables (set them to $s, t \in \mathbb{R}$)

$$\begin{aligned} x_1 &= -28 - 3s - 4t \\ x_2 &= s \\ x_3 &= -14 - 2t \\ x_4 &= t \\ x_5 &= 5 \end{aligned} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -28 \\ 0 \\ -14 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} t$$

Summary

Let $A\mathbf{x} = \mathbf{b}$ be a general linear system in n variables and m equations:

- If $\text{rank}(A) = r < m$ and $\text{rank}(A \mid b) = r + 1$ then the system is inconsistent. (the row echelon form of the augmented matrix has a row $[0\ 0\ \dots\ 0\ 1]$)

- If $\text{rank}(A) = r = \text{rank}(A \mid \mathbf{b})$ then the system is consistent and there are $n - r$ free variables;
  if $r < n$ there are infinitely many solutions, if $r = n$ there are no free variables and the solution is unique

Let $A\mathbf{x} = \mathbf{0}$ be an homogeneous system in n variables and m equations, $\text{rank}(A) = r$ (always consistent):

- if $r < n$ there are infinitely many solutions, if $r = n$ there are no free variables and the solution is unique, $\mathbf{x} = \mathbf{0}$.

General solutions in vector notation

Example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -28 \\ 0 \\ -14 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} t, \quad \forall s, t \in \mathbb{R}$$

For $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} = \mathbf{p} + \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-r} \mathbf{v}_{n-r}, \quad \forall \alpha_i \in \mathbb{R}, i = 1, \ldots, n-r$$

Note:

- if $\alpha_i = 0, \forall i = 1, \ldots, n-r$ then $A\mathbf{p} = \mathbf{b}$, ie, $\mathbf{p}$ is a particular solution
- if $\alpha_1 = 1$ and $\alpha_i = 0, \forall i = 2, \ldots, n-r$ then

$$A(\mathbf{p} + \mathbf{v}_1) = \mathbf{b} \;\rightarrow\; A\mathbf{p} + A\mathbf{v}_1 = \mathbf{b} \;\xrightarrow{A\mathbf{p}=\mathbf{b}}\; A\mathbf{v}_1 = 0$$

Thus (recall that $\mathbf{x} = \mathbf{p} + \mathbf{z}$, $\mathbf{z} \in N(A)$):

- If A is an $m \times n$ matrix of rank r, the general solutions of $A\mathbf{x} = \mathbf{b}$ is the sum of:
  - a particular solution $\mathbf{p}$ of the system $A\mathbf{x} = \mathbf{b}$ and
  - a linear combination $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-r}\mathbf{v}_{n-r}$ of solutions $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{n-r}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$

- If A has rank n, then $A\mathbf{x} = \mathbf{0}$ only has the solution $\mathbf{x} = \mathbf{0}$ and so $A\mathbf{x} = \mathbf{b}$ has a unique solution: $\mathbf{p}$

Outline

Range

Definition (Range of a matrix)

Let A be an $m \times n$ matrix, the range of A, denoted by $R(A)$, is the subset of $\mathbb{R}^m$ given by

$$R(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

That is, the range is the set of all vectors $\mathbf{y} \in \mathbb{R}^m$ of the form $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, or
all $\mathbf{y} \in \mathbb{R}^m$ for which the system $A\mathbf{x} = \mathbf{y}$ is consistent.

Recall, if $\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ is any vector in $\mathbb{R}^n$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \qquad i = 1, \ldots, n.$$

Then $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and

$$A\mathbf{x} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_n\mathbf{a}_n$$

that is, vector $A\mathbf{x}$ in $\mathbb{R}^n$ as a linear combination of the column vectors of A
Proof?

Hence $R(A)$ is the set of all linear combinations of the columns of A.
$\rightsquigarrow$ the range is also called the column space of A:

$$R(A) = \{\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_n\mathbf{a}_n \mid \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\}$$

Thus, $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is in the range of A, ie, a linear combination of the columns of A

Example

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

Then, for $\mathbf{x} = [\alpha_1, \alpha_2]^T$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \alpha_1 + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \alpha_2$$

so

$$R(A) = \left\{ \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} \middle| \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

Example

$$\begin{cases} \quad x + 2y = 0 \\ -x + 3y = -5 \\ 2x + y = 3 \end{cases}$$

$$\begin{cases} \quad x + 2y = 1 \\ -x + 3y = -5 \\ 2x + y = 2 \end{cases}$$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\mathbf{a}_1 - \mathbf{a}_2$$

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$. (Why?) Only way:

$$0 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0\mathbf{a}_1 + 0\mathbf{a}_2 = \mathbf{0}$$

Hence no way to express $[1, -5, 2]$ as linear expression of the two columns of A.

Outline

1. Rank

2. Range

3. Vector Spaces

Premise

- We move to a higher level of abstraction
- A vector space is a set with an addition and scalar multiplication that behave appropriately, that is, like $\mathbb{R}^n$
- Imagine a vector space as a class of a generic type (template) in object oriented programming, equipped with two operations.

Vector Spaces

Definition (Vector Space)

A (real) vector space V is a non-empty set equipped with an addition and a scalar multiplication operation such that for all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$:

1. $\mathbf{u} + \mathbf{v} \in V$ (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative law for addition)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associative law for addition)
4. there is a single member $\mathbf{0}$ of V, called the zero vector, such that for all $\mathbf{v} \in V, \mathbf{v} + \mathbf{0} = \mathbf{v}$
5. for every $\mathbf{v} \in V$ there is an element $\mathbf{w} \in V$, written $-\mathbf{v}$, called the negative of $\mathbf{v}$, such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
6. $\alpha\mathbf{v} \in V$ (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (distributive law)
8. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ (distributive law)
9. $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ (associative law for vector multiplication)
10. $1\mathbf{v} = \mathbf{v}$

Examples

- set $\mathbb{R}^n$
- but the set of objects for which the vector space defined is valid are more than the vectors in $\mathbb{R}^n$.
- set of all functions $F : \mathbb{R} \to \mathbb{R}$.
  We can define an addition $f + g$:

  $$(f + g)(x) = f(x) + g(x)$$

  and a scalar multiplication αf:

  $$(\alpha f)(x) = \alpha f(x)$$

- Example: $x + x^2$ and $2x$. They can represent the result of the two operations.
- What is $-f$? and the zero vector?

The axioms given are minimum number needed.
Other properties can be derived:
For example:

$$(-1)\mathbf{x} = -\mathbf{x}$$

$$\mathbf{0} = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = \mathbf{x} + (-1)\mathbf{x}$$

Adding $-\mathbf{x}$ on both sides:

$$-\mathbf{x} = -\mathbf{x} - \mathbf{0} = -\mathbf{x} + \mathbf{x} + (-1)\mathbf{x} = (-1)\mathbf{x}$$

which proves that $-\mathbf{x} = (-1)\mathbf{x}$.

Try the same with $-f$.

Examples

- $V = \{\mathbf{0}\}$

- the set of $m \times n$ all matrices

- the set of all infinite sequences of real numbers,
  $\mathbf{y} = \{y_1, y_2, \ldots, y_n, \ldots,\}, y_i \in \mathbb{R}$. ($\mathbf{y} = \{y_n\}$, $n \geq 1$)
  addition of $\mathbf{y} = \{y_1, y_2, \ldots, y_n, \ldots,\}$ and $\mathbf{z} = \{z_1, z_2, \ldots, z_n, \ldots,\}$ then:
  $$\mathbf{y} + \mathbf{z} = \{y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n, \ldots,\}$$
  multiplication by a scalar $\alpha \in \mathbb{R}$:
  $$\alpha\mathbf{y} = \{\alpha y_1, \alpha y_2, \ldots, \alpha y_n, \ldots,\}$$

- set of all vectors in $\mathbb{R}^3$ with the third entry equal to 0 (verify closure):
  $$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

Linear Combinations

Definition (Linear Combination)

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V, the vector

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k$$

is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
The scalars α_i are called coefficients.

- To find the coefficients that given a set of vertices express by linear combination a given vector, we solve a system of linear equations.
- If F is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ then the function $f : x \mapsto 2x^2 + 3x + 4$ can be expressed as a linear combination of:

  $$f = 2g + 3h + 4k$$

  where $g : x \mapsto x^2$, $h : x \mapsto x$, $k : x \mapsto 1$
- Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, is it possible to represent any point in the Cartesian plane?

Subspaces

Definition (Subspace)

A subspace W of a vector space V is a non-empty subset of V that is itself a vector space under the same operations of addition and scalar multiplication as V.

Theorem

Let V be a vector space. Then a non-empty subset W of V is a subspace if and only if both the following hold:

- *for all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$*
  (W is closed under addition)
- *for all $\mathbf{v} \in W$ and $\alpha \in \mathbb{R}$, $\alpha\mathbf{v} \in W$*
  (W is closed under scalar multiplication)

ie, all other axioms can be derived to hold true

Example

- The set of all vectors in $\mathbb{R}^3$ with the third entry equal to 0.
- The set $\{\mathbf{0}\}$ is not empty, it is a subspace since $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha\mathbf{0} = \mathbf{0}$ for any $\alpha \in \mathbb{R}$.

Example

In $\mathbb{R}^2$, the lines $y = 2x$ and $y = 2x + 1$ can be defined as the sets of vectors:

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = 2x, x \in \mathbb{R} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = 2x + 1, x \in \mathbb{R} \right\}$$

$$S = \{\mathbf{x} \mid \mathbf{x} = t\mathbf{v}, t \in \mathbb{R}\} \qquad U = \{\mathbf{x} \mid \mathbf{x} = \mathbf{p} + t\mathbf{v}, t \in \mathbb{R}\}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example (cntd)

1. The set S is non-empty, since $\mathbf{0} = 0\mathbf{v} \in S$.
2. closure under addition:

$$\mathbf{u} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S, \quad \mathbf{w} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S, \qquad \text{for some } s, t \in \mathbb{R}$$

$\mathbf{u} + \mathbf{w} = s\mathbf{v} + t\mathbf{v} = (s+t)\mathbf{v} \in S$ since $s + t \in \mathbb{R}$

3. closure under scalar multiplication:

$$\mathbf{u} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S \qquad \text{for some } s \in \mathbb{R}, \qquad \alpha \in \mathbb{R}$$

$\alpha\mathbf{u} = \alpha(s(\mathbf{v})) = (\alpha s)\mathbf{v} \in S$ since $\alpha s \in \mathbb{R}$

Note that:

- $\mathbf{u}, \mathbf{w}$ and $\alpha \in \mathbb{R}$ must be arbitrary

Example (cntd)

1. $\mathbf{0} \notin U$
2. U is not closed under addition:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in U, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in U \quad \text{but} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \notin U$$

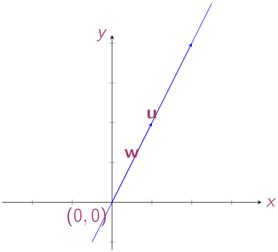
3. U is not closed under scalar multiplication

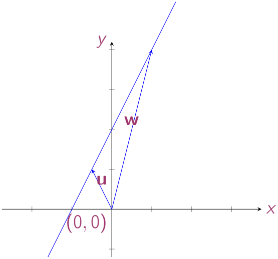
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in U, 2 \in \mathbb{R} \quad \text{but} \quad 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin U$$

Note that:

- proving just one of the above couterexamples is enough to show that U is not a subspace
- it is sufficient to make them fail for particular choices
- a good place to start is checking whether $\mathbf{0} \in S$. If not then S is not a subspace

Geometric interpretation:





⇝ The line $y = 2x + 1$ is an affine subset, a „translation“ of a subspace

Theorem

A non-empty subset W of a vector space is a subspace if and only if for all $\mathbf{u}, \mathbf{v} \in W$ and all $\alpha, \beta \in \mathbb{R}$, we have $\alpha\mathbf{u} + \beta\mathbf{v} \in W$.
That is, W is closed under linear combination.

Summary

- Rank of a matrix and relation to number of solutions of a linear system
- General solutions of a linear system in vector notation
- Range, set of linear combinations of the columns of a matrix
- Vector spaces: properties
- Linear combination
- Subspaces: non-empty + closed under linear combination