DM559

Linear and Integer Programming

Lecture 7
Vector Spaces (cntd)
Linear Independence, Bases and Dimension

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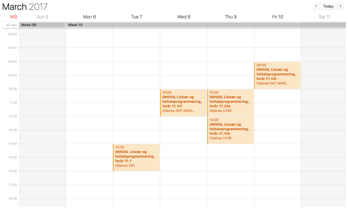
Outlook

Week 9:

Section H1:	Section H2:
Today, Introductory class	Today, Introductory class
Tomorrow, 14-16, Applications	Tomorrow, 12-14, Introductory class
Thursday, 12-14, Introductory class	**Tomorrow, 14-16, Applications**

Outlook



<u>Week 10:</u>

Section H1:	Section H2:
Tuesday, 14-16, Exercise class	Tuesday, 14-16, Exercise class
Wednesday, 10-12, Laboratory class	Thursday, 10-12, Exercise class
Thursday, 12-14, Exercise class	Friday, 08-10, Laboratory class

a) Join? H1, Thursday, 12-14 $\Longleftrightarrow$ H2, Thursday, 10-12

1. Move H1 from Thursday, 12-14 to Thursday, 10-12?
2. Move H2 from Thursday, 10-12 to Thursday, 12-14? ✔

b) Join? H1, Wednesday, 10-12 $\Longleftrightarrow$ H2, Friday, 08-10

1. Move H1 from Wednesday, 10-12 to Friday, 08-10?
2. Move H2 from Friday, 08-10 to Wednesday, 10-12? ✔

Resume

- Matrix Calculus
- Geometric Insight
- Systems of Linear Equations. Gaussian Elimination.
- Elementary Matrices, Determinants, Matrix Inverse.

Last Time:

- Rank (number of leading ones in REF)
  Relationship with linear systems
- (Numerical methods, LU + iterative)
- Range of a matrix
- Vector Spaces: Definition, Examples. Linear combination.
- Subspaces

Outline

1. Vector Spaces (cntd)
2. Linear independence
3. Bases
4. Dimension

Outline

Linear Span

- If $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_k\mathbf{v}_k$, then $\mathbf{v} + \mathbf{w}$ and $s\mathbf{v}, s \in \mathbb{R}$ are also linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
- The set of all linear combinations of a given set of vectors of a **vector space** V forms a **subspace**:

Definition (Linear span)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. The linear span of $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, denoted by $\text{Lin}(X)$, that is:

$$\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}$$

Theorem

If $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors of a vector space V, then $\text{Lin}(X)$ is a ***subspace*** *of V and is also called the* ***subspace*** *spanned by X.*
It is the smallest **subspace** *containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.*

Example

- $\text{Lin}(\{\mathbf{v}\}) = \{\alpha\mathbf{v} \mid \alpha \in \mathbb{R}\}$ defines a line in $\mathbb{R}^n$.
- Recall that a plane in $\mathbb{R}^3$ has two equivalent representations:

$$ax + by + cz = d \qquad \text{and} \qquad \mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

where $\mathbf{v}$ and $\mathbf{w}$ are non parallel.

  - If $d = 0$ and $\mathbf{p} = \mathbf{0}$, then

    $$\{\mathbf{x} \mid \mathbf{x} = s\mathbf{v} + t\mathbf{w}, s, t, \in \mathbb{R}\} = \text{Lin}(\{\mathbf{v}, \mathbf{w}\})$$

    and hence a **subspace** of $\mathbb{R}^n$.

  - If $d \neq 0$, then the plane is not a **subspace**. It is an affine subset, a translation of a **subspace**.

    (recall that one can also show directly that a subset is a **subspace** or not)

Outline

Linear Independence

Definition (Linear Independence)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent (or form a linearly independent set) if and only if the vector equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

has the unique solution

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Definition (Linear Dependence)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent (or form a linearly dependent set) if and only if there are real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

Example

In $\mathbb{R}^2$, the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are linearly independent. Indeed:

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} \alpha + \beta = 0 \\ 2\alpha - \beta = 0 \end{cases}$$

The homogeneous linear system has only the trivial solution, $\alpha = 0, \beta = 0$, so linear independence.

Example

In $\mathbb{R}^3$, the following vectors are linearly dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

Indeed: $2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$

Theorem

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$ is linearly dependent if and only if at least one vector $\mathbf{v}_i$ is a linear combination of the other vectors.

Proof

$\Longrightarrow$

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are linearly dependent then

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

has a solution with some $\alpha_i \neq 0$, then:

$$\mathbf{v}_i = -\frac{\alpha_1}{\alpha_i}\mathbf{v}_1 - \frac{\alpha_2}{\alpha_i}\mathbf{v}_2 - \cdots - \frac{\alpha_{i-1}}{\alpha_i}\mathbf{v}_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}\mathbf{v}_{i+1} + \cdots - \frac{\alpha_k}{\alpha_i}\mathbf{v}_k$$

which is a linear combination of the other vectors

$\Longleftarrow$

If $\mathbf{v}_i$ is a lin combination of the other vectors, eg,

$$\mathbf{v}_i = \beta_1 \mathbf{v}_1 + \cdots + \beta_{i-1}\mathbf{v}_{i-1} + \beta_{i+1}\mathbf{v}_{i+1} + \cdots + \beta_k \mathbf{v}_k$$

then

$$\beta_1 \mathbf{v}_1 + \cdots + \beta_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + \beta_{i+1}\mathbf{v}_{i+1} + \cdots + \beta_k \mathbf{v}_k = \mathbf{0}$$

□

Corollary

Two vectors are linearly dependent if and only if at least one vector is a scalar multiple of the other.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

are linearly independent

Theorem

In a **vector space** V*, a non-empty set of vectors that contains the zero vector is linearly dependent.*

Proof:

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset V$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{0}\}$$

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_k + a\mathbf{0} = \mathbf{0}, \qquad a \neq 0$$

Uniqueness of linear combinations

Theorem

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *are linearly independent vectors in* V *and if*

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_k\mathbf{v}_k$$

then

$$a_1 = b_1, \quad a_2 = b_2, \quad \ldots \quad a_k = b_k.$$

- If a vector $\mathbf{x}$ can be expressed as a linear combination of linearly independent vectors, then this can be done in only one way

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

Testing for Linear Independence in $\mathbb{R}^n$

For k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$$

is equivalent to

$$A\mathbf{x}$$

where A is the $n \times k$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and $\mathbf{x} = [\alpha_1, \alpha_2, \ldots, \alpha_k]^T$:

Theorem

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly dependent if and only if the linear system $A\mathbf{x} = \mathbf{0}$, where A is the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_k]$, has a solution other than $\mathbf{x} = \mathbf{0}$.
Equivalently, the vectors are linearly independent precisely when the only solution to the system is $\mathbf{x} = \mathbf{0}$.

If vectors are linearly dependent, then any solution $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} = [\alpha_1, \alpha_2 \ldots, \alpha_k]^T$ of $A\mathbf{x} = \mathbf{0}$ gives a non-trivial linear combination $A\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k = \mathbf{0}$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

are linearly dependent.
We solve $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -5 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

The general solution is

$$\mathbf{v} = \begin{bmatrix} t \\ -3t \\ t \end{bmatrix}$$

and $A\mathbf{x} = t\mathbf{v}_1 - 3t\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$

Hence, for $t = 1$ we have: $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Recall that $A\mathbf{x} = \mathbf{0}$ has precisely one solution $\mathbf{x} = \mathbf{0}$ iff the $n \times k$ matrix is row equiv. to a row echelon matrix with k leading ones, ie, iff $\text{rank}(A) = k$

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent iff the $n \times k$ matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_k]$ has rank k.

Theorem

The maximum size of a linearly independent set of vectors in $\mathbb{R}^n$ is n.

- $\text{rank}(A) \leq \min\{n, k\}$, hence $\text{rank}(A) \leq n \Rightarrow$ when lin. indep. $k \leq n$.
- we exhibit an example that has exactly n independent vectors in $\mathbb{R}^n$ (there are infinite examples):

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This is known as the standard basis of $\mathbb{R}^n$.

Example

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 1 \end{bmatrix} \right\} \quad \text{lin. dep.} \quad \text{since } 5 > n = 4$$

$$L_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix} \right\} \quad \text{lin. indep.}$$

$$L_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\} \quad \text{lin. dep. since } \operatorname{rank}(A) = 2$$

$$L_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{lin. dep. since } L_3 \subseteq L_4$$

Linear Independence and Span in $\mathbb{R}^n$

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.
What are the conditions for S to span $\mathbb{R}^n$ and be linearly independent?

Let A be the $n \times k$ matrix whose columns are the vectors from S.

- S spans $\mathbb{R}^n$ if for any $v \in \mathbb{R}^n$ the linear system $A\mathbf{x} = \mathbf{v}$ is consistent. This happens when $\text{rank}(A) = n$, hence $k \geq n$
- S is linearly independent iff the linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution. This happens when $\text{rank}(A) = k$, Hence $k \leq n$

Hence, to span $\mathbb{R}^n$ and to be linearly independent, the set S must have exactly n vectors and the square matrix A must have $\det(A) \neq 0$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 5 & 1 \end{vmatrix} = 30 \neq 0$$

Outline

Bases

Definition (Basis)

Let V be a **vector space**. Then the subset $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of V is said to be a basis for V if:

1. B is a linearly independent set of vectors, and
2. B spans V; that is, $V = \text{Lin}(B)$

Theorem

*If V is a **vector space**, then a smallest spanning set is a basis of V.*

Theorem

$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V if and only if any $\mathbf{v} \in V$ is a unique linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Example

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.
the vectors are linearly independent and for any $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$,
$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$, ie,

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example

The set below is a basis of $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

- any vector $\mathbf{x} \in \mathbb{R}^2$ can be written as a linear combination of vectors in S.
- any vector $\mathbf{b}$ is a linear combination of the two vectors in S
  $\rightsquigarrow A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$.
- S spans $\mathbb{R}^2$ and is linearly independent

Example

Find a basis of the **subspace** of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y - 3z = 0 \right\}.$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x + 3z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = x\mathbf{v} + z\mathbf{w}, \quad \forall x, z \in \mathbb{R}$$

The set $\{\mathbf{v}, \mathbf{w}\}$ spans W. The set is also independent:

$$\alpha\mathbf{v} + \beta\mathbf{w} = \mathbf{0} \implies \alpha = 0, \beta = 0$$

Coordinates

Definition (Coordinates)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V, then any vector $\mathbf{v} \in V$ can be expressed uniquely as $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$ then the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coordinates of $\mathbf{v}$ with respect to the basis S. We use the notation

$$[\mathbf{v}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_S$$

to denote the coordinate vector of $\mathbf{v}$ in the basis S.

Example

Consider the two basis of $\mathbb{R}^2$:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}_B \qquad [\mathbf{v}]_S = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_S$$

In the standard basis the coordinates of $\mathbf{v}$ are precisely the components of the vector $\mathbf{v}$.
In the basis S, they are such that

$$\mathbf{v} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Extension of the main theorem

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. *A is invertible*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}$*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$*
4. *the reduced row echelon form of A is I.*
5. *$|A| \neq 0$*
6. *The rank of A is n*
7. *The column vectors of A are a basis of $\mathbb{R}^n$*
8. *The rows of A (written as vectors) are a basis of $\mathbb{R}^n$*

(The last statement derives from $|A^T| = |A|$.)
Hence, simply calculating the determinant can inform on all the above facts.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

This set is linearly dependent since $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$
so $\mathbf{v}_3 \in \mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\})$ and $\mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.
The linear span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^3$ is a plane:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2 = s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The vector $\mathbf{x}$ belongs to the **subspace** iff it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, that is, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}$ are linearly dependent or:

$$|A| = \begin{vmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 3 & 5 & z \end{vmatrix} = 0 \qquad \Longrightarrow \qquad |A| = 7x + y - 3z = 0$$

Outline

1. Vector Spaces (cntd)

2. Linear independence

3. Bases

4. Dimension

Dimension

Theorem

Let V be a **vector space** *with a basis*

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

of n vectors. Then any set of $n+1$ vectors is linearly dependent.

Proof:
Omitted (choose an arbitrary set of $n+1$ vectors in V and show that since any of them is spanned by the basis then the set must be linearly dependent.)

It follows that:

Theorem

Let a **vector space** V *have a finite basis consisting of* r *vectors. Then any basis of* V *consists of exactly* r *vectors.*

Definition (Dimension)

The number of k vectors in a finite basis of a **vector space** V is the dimension of V and is denoted by $\dim(V)$.
The **vector space** $V = \{\mathbf{0}\}$ is defined to have dimension 0.

- a plane in $\mathbb{R}^2$ is a two-dimensional **subspace**
- a line in $\mathbb{R}^n$ is a one-dimensional **subspace**
- a hyperplane in $\mathbb{R}^n$ is an $(n-1)$-dimensional **subspace** of $\mathbb{R}^n$
- the **vector space** F of real functions is an infinite-dimensional **vector space**
- the **vector space** of real-valued sequences is an infinite-dimensional **vector space**.

Dimension and bases of Subspaces

Example

The plane W in $\mathbb{R}^3$

$$W = \{\mathbf{x} \mid x + y - 3z = 0\}$$

has a basis consisting of the vectors $\mathbf{v}_1 = [1, 2, 1]^T$ and $\mathbf{v}_2 = [3, 0, 1]^T$.

Let $\mathbf{v}_3$ be any vector $\notin W$, eg, $\mathbf{v}_3 = [1, 0, 0]^T$. Then the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

Basis of a Linear Space

If we are given k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, how can we find a basis for $\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\})$?

We can:

- create an $n \times k$ matrix (vectors as columns) and find a basis for the column space by putting the matrix in reduced row echelon form

Summary

- Linear dependence and independence
- Determine linear dependency of a set of vectors, ie, find non-trivial lin. combination that equal zero
- Basis
- Find a basis for a linear space
- Dimension (finite, infinite)