DM841
DISCRETE OPTIMIZATION

Global constraints (2/2)

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Outline

1. Global Constraints

Outline

Global Constraint: among and sequence

among

Let $x_1, \ldots, x_n$ be a tuple of variables, S a set of variables, and l and u two nonnegative integers

among$([x_1, ..., x_n], S, l, u)$

At least l and at most u of variables take values in S.
In Gecode: count

sequence

Let $x_1, \ldots, x_n$ be a tuple of variables, S a set of variables, and l and u two nonnegative integers, s a positive integer.

sequence$([x_1, ..., x_n], S, l, u, s)$

At least l and at most u of variables take values from S in s consecutive variables

Car Sequencing Problem

Car Sequencing Problem

- an assembly line makes 50 cars a day
- 4 types of cars
- each car type is defined by options: {air conditioning, sun roof}

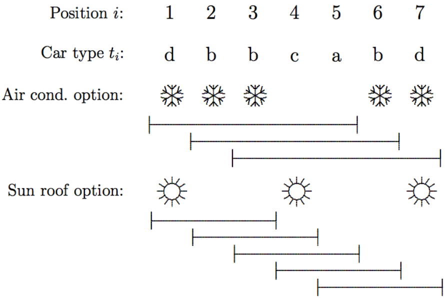
type	air cond.	sun roof	demand
a	no	no	20
b	yes	no	15
c	no	yes	8
d	yes	yes	7

- at most 3 cars in any sequence of 5 can be given air conditioning
- at most 1 in any sequence of 3 can be given a sun roof

Task: sequence the car types so as to meet demands while observing capacity constraints of the assembly line.

Car Sequencing Problem

Sequence constraints



Car Sequencing Problem: CP model

Car Sequencing Problem

Let t_i be the decision variable that indicates the type of car to assign to each position i in the sequence.

$$
\begin{aligned}
&\text{cardinality}([t_1, \ldots, t_{50}], (a, b, c, d), (20, 15, 8, 7), (20, 15, 8, 7)) \\
&\text{among}([t_i, \ldots, t_{i+4}], \{b, d\}, 0, 3), \qquad \forall i = 1..46 \\
&\text{among}([t_i, \ldots, t_{i+2}], \{c, d\}, 0, 1), \qquad \forall i = 1..48 \\
&t_i \in \{a, b, c, d\}, i = 1, \ldots, 50.
\end{aligned}
$$

Note: in Gecode `among` is `count`.
However, we can use `sequence` for the two `among` constraints above:

$$
\begin{aligned}
&\text{sequence}([t_1, \ldots, t_{50}], \{b, d\}, 0, 3, 5), \\
&\text{sequence}([t_1, \ldots, t_{50}], \{c, d\}, 0, 1, 3),
\end{aligned}
$$

Car Sequencing Problem: MIP model

$$\begin{pmatrix} AC_i = 0 \\ SR_i = 0 \end{pmatrix} \vee \begin{pmatrix} AC_i = 1 \\ SR_i = 0 \end{pmatrix} \vee \begin{pmatrix} AC_i = 0 \\ SR_i = 1 \end{pmatrix} \vee \begin{pmatrix} AC_i = 1 \\ SR_i = 1 \end{pmatrix}$$

$$AC_i = AC_i^a + AC_i^b + AC_i^c + AC_i^d$$

$$SR_i = SR_i^a + SR_i^b + SR_i^c + SR_i^d$$

$$AC_i^a = 0, \quad AC_i^b = \delta_{ib}, \quad AC_i^c = 0, \quad AC_i^d = \delta_{id}$$

$$SR_i^a = 0, \quad SR_i^b = 0, \quad SR_i^c = \delta_{ic}, \quad SR_i^d = \delta_{id}$$

$$\delta_{ia} + \delta_{ib} + \delta_{ic} + \delta_{id} = 1$$

$$\delta_{ij} \in \{0, 1\}, \quad j = a, b, c, d$$

$$AC_i = \delta_{ib} + \delta_{id}, \quad SR_i = \delta_{ic} + \delta_{id}, \quad i = 1, \ldots, 50$$

$$\delta_{ib} + \delta_{ic} + \delta_{id} \leq 1, \quad i = 1, \ldots, 50$$

$$\delta_{ij} \in \{0, 1\}, \quad j = b, c, d, \quad i = 1, \ldots, 50$$

$$\sum_{i=1}^{50} \delta_{ia} = 20, \quad \sum_{i=1}^{50} \delta_{ib} = 15, \quad \sum_{i=1}^{50} \delta_{ic} = 8, \quad \sum_{i=1}^{50} \delta_{id} = 7, \quad i = 1, \ldots, 50$$

$$\sum_{j=i}^{i+4} AC_j \leq 3, \quad i = 1, \ldots, 46$$

$$\sum_{j=j}^{i+2} SR_j \leq 1, \quad j = 1, \ldots, 48$$

Global Constraint: nvalues

nvalues

Let $x_1, \ldots, x_n$ be a tuple of variables, and l and u two nonnegative integers

$$\texttt{nvalues}([x_1, ..., x_n], l, u)$$

At least l and at most u different values among the variables

⇝ generalization of `alldifferent`

In Gecode: `nvalues`

Global Constraint: `stretch`

`stretch` (In Gecode: via `regular` and `extensional`)

Let $x_1, \ldots, x_n$ be a tuple of variables with finite domains,
v an m-tuple of possible values of the variables,
l an m-tuple of lower bounds and u an m-tuple of upper bounds.
A stretch is a maximal sequence of consecutive variables that take the same value, i.e., $x_j, \ldots, x_k$ for v if $x_j = \ldots = x_k = v$ and $x_{j-1} \neq v$ (or $j = 1$) and $x_{k+1} \neq v$ (or $k = n$).

$$\texttt{stretch}([x_1, ..., x_n], \mathbf{v}, \mathbf{l}, \mathbf{u}) \qquad \texttt{stretch-cycle}([x_1, ..., x_n], \mathbf{v}, \mathbf{l}, \mathbf{u})$$

for each $j \in \{1, \ldots, m\}$ any stretch of value v_j in x have length at least l_j and at most u_j.

In addition:

$$\texttt{stretch}([x_1, ..., x_n], \mathbf{v}, \mathbf{l}, \mathbf{u}, P)$$

with P set of patterns, i.e., pairs $(v_j, v_{j'})$. It imposes that a stretch of values v_j must be followed by a stretch of value $v_{j'}$

Global Constraint: element

"element" constraint

Let y be an integer variable,
z a variable with finite domain,
and c an array of constants, i.e., $c = [c_1, c_2, \ldots, c_n]$.
The element constraint states that z is equal to the y-th variable in c, or $z = c_y$.
More formally:

$$\texttt{element}(y, z, [c_1, \ldots, c_n]) = \{(e, f) \mid e \in D(y), f \in D(z), f = c_e\}.$$

```
IntArgs c(5, 1,4,9,16,25);
element(home, c, x, y);
```

Assignment problems

The assignment problem is to find a minimum cost assignment of m tasks to n workers ($m \leq n$).
Each task is assigned to a different worker, and no two workers are assigned the same task.
If assigning worker i to task j incurs cost c_{ij}, the problem is simply stated:

$$\min \sum_{i=1,\ldots,n} c_{ix_i}$$

$$\texttt{alldiff}([x_1, \ldots, x_n]),$$

$$x_i \in D_i, \forall i = 1, \ldots, n.$$

Note: cost depends on position. Recall: with $n = m$ min weighted bipartite matching (Hungarian method)
with supplies/demands transshipment problem

Global Constraint: channel

"channel" constraint

Let x be an array of boolean variables and y be an integer variable:

$$\texttt{channel}([x_1, \ldots, x_n], y) =$$
$$\{([e_1, \ldots, e_n], d) \mid e_i \in \{0,1\}, \forall i, d \in D(y), \forall j, e_i = 1 \iff d = i\}.$$

"channel" constraint

Let y be array of integer variables, and x be an array of integer variables:

$$\texttt{channel}([y_1, \ldots, y_n], [x_1, \ldots, x_n]) =$$
$$\{([e_1, \ldots, e_n], [d_1, \ldots, d_n]) \mid e_i \in D(y_i), \forall i, d_j \in D(x_j), \forall j, e_i = j \wedge d_j = i\}.$$

Employee Scheduling problem

Four nurses are to be assigned to eight-hour shifts.
Shift 1 is the daytime shift, while shifts 2 and 3 occur at night.
The schedule repeats itself every week. In addition,

1. Every shift is assigned exactly one nurse.
2. Each nurse works at most one shift a day.
3. Each nurse works at least five days a week.
4. To ensure a certain amount of continuity, no shift can be staffed by more than two different nurses in a week.
5. To avoid excessive disruption of sleep patterns, a nurse cannot work different shifts on two consecutive days.
6. Also, a nurse who works shift 2 or 3 must do so at least two days in a row.

Employee Scheduling problem

Feasible Solutions

Solution viewed as assigning workers to shifts.

	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Shift1	A	B	A	A	A	A	A
Shift2	C	C	C	B	B	B	B
Shift3	D	D	D	D	C	C	D

Solution viewed as assigning shifts to workers.

	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Worker A	1	0	1	1	1	1	1
Worker B	0	1	0	2	2	2	2
Worker C	2	2	2	0	3	3	0
Worker D	3	3	3	3	0	0	3

Employee Scheduling problem

Feasible Solutions

Let w_{sd} be the nurse assigned to shift s on day d, where the domain of w_{sd} is the set of nurses $\{A, B, C, D\}$.
Let t_{id} be the shift assigned to nurse i on day d, and where shift 0 denotes a day off.

1. alldiff$(w_{1d}, w_{2d}, w_{3d}), d = 1, \ldots, 7$
2. cardinality$(W, (A, B, C, D), (5, 5, 5, 5), (6, 6, 6, 6))$
3. nvalues$(\{w_{s1}, \ldots, w_{s7}\}, 1, 2), s = 1, 2, 3$
4. alldiff$(t_{Ad}, t_{Bd}, t_{Cd}, t_{Dd}), d = 1, ..., 7$
5. cardinality$(\{t_{i1}, \ldots, t_{i7}\}, 0, 1, 2), i = A, B, C, D$
6. stretch-cycle$((t_{i1}, \ldots, t_{i7}), (2, 3), (2, 2), (6, 6), P), i = A, B, C, D$
7. $w_{t_{id} d} = i, \forall i, d, \quad t_{w_{sd} s} = s, \forall s, d$

Circuit problems

Given a directed weighted graph $G = (N, A)$, find a circuit of min cost:

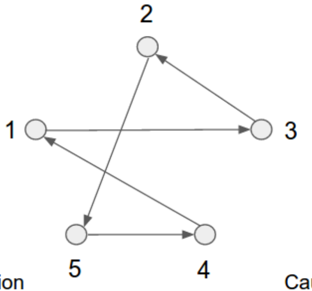
$$\begin{aligned} \min \quad & \sum_{i=1,\ldots,n} c_{x_i x_{i+1}} \\ & \texttt{alldiff}([x_1, \ldots, x_n]), \\ & x_i \in D_i, \forall i = 1, \ldots, n. \end{aligned}$$

Note: cost depends on sequence.

An alternative formulation is

$$\begin{aligned} \min \quad & \sum_{i=1,\ldots,n} c_{i y_i} \\ & \texttt{circuit}([y_1, \ldots, y_n]), \\ & y_i \in D_i = \{j \mid (i, j) \in A\}, \forall i = 1, \ldots, n. \end{aligned}$$

Circuit representation



Classical permutation notation

	1	2	3	4	5
x:	1	3	2	5	4

Cauchy two-line notation

	1	2	3	4	5
y:	3	5	2	1	4

$$Y[\ x[i]\ ] = x[i+1]$$

Global Constraint: `circuit`

"`circuit`" constraint

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of variables with respective domains $D(x_i) \subseteq \{1, 2, ..., n\}$ for $i = 1, 2, ..., n$. Then

$$\texttt{circuit}(x_1, \ldots, x_n) = \{(d_1, ..., d_n) \mid \forall i, d_i \in D(x_i), d_1, ..., d_n \text{ is cyclic } \}.$$

Circuit problems - Linking viewpoints

A model with redundant constraints is as follows:

$$\min \quad z \tag{1}$$

$$z \geq \sum_{i=1,\ldots,n} c_{x_i x_{i+1}} \tag{2}$$

$$z \geq \sum_{i=1,\ldots,n} c_{i y_i} \tag{3}$$

$$\texttt{alldiff}([x_1, \ldots, x_n]), \tag{4}$$

$$\texttt{circuit}([y_1, \ldots, y_n]), \tag{5}$$

$$x_1 = y_{x_n} = 1, \quad x_{i+1} = y_{x_i}, i = 1, \ldots, n-1 \tag{6}$$

$$x_i \in \{1, \ldots, n\}, \forall i = 1, \ldots, n, \tag{7}$$

$$y_i \in D_i = \{j \mid (i,j) \in A\}, \forall i = 1, \ldots, n. \tag{8}$$

Line (6) implements the linking between the two formulations.
In Gecode it can be implemented with the `element`:

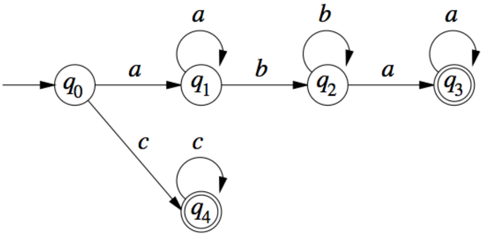
```
element(y, x[i], x[i+1])
```

Global Constraint: regular

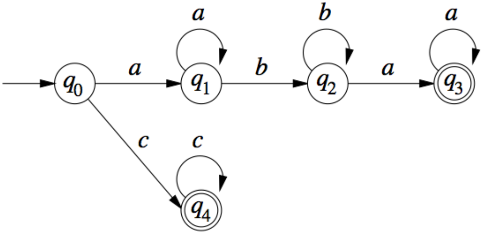
"regular" constraint

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of variables with $D(x_i) \subseteq \Sigma$ for $1 \leq i \leq n$. Then

$$\texttt{regular}(X, M) = \{(d_1, ..., d_n) \mid \forall i, d_i \in D(x_i), [d_1, d_2, \ldots, d_n] \in L(M)\}.$$



Global Constraint: regular



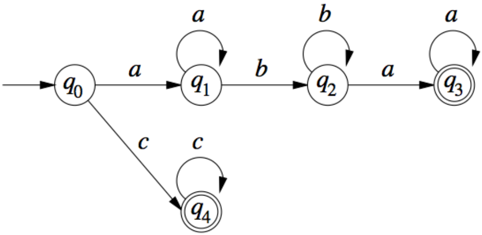
Example

Given the problem

$$x_1 \in \{a, b, c\}, \quad x_2 \in \{a, b, c\}, \quad x_3 \in \{a, b, c\}, \quad x_4 \in \{a, b, c\},$$

$$\texttt{regular}([x_1, x_2, x_3, x_4], M).$$

One solution to this CSP is $x_1 = a, x_2 = b, x_3 = a, x_4 = a$.



In Gecode:

```
DFA::Transition t[] = {{0, 0(a), 1}, {1, 0(a), 1}, {1, 1(b), 2}, {2, 1(b), 2},
                       {2, 0(a), 3}, {3, 0(a), 3}, {3, 0, -1},
                       {0,2(c),4}, {4, 2(c), 4}, {4, 0, -1}};
int f[] = {3,4,-1}; // vector of final states
DFA d(0, t, f);
BoolVarArray x(home, 4, 0(a), 3(d));
extensional(home, x, d);
```

```
REG r=(REG(0) + *REG(0) + REG(1) + *REG(1) + REG(0) + *REG(0)) | REG(2) + (*REG(2)));
DFA d(r);
extensional(home, x, d);
```

Scheduling Constraints

One job at a time on a machine (disjunctive machines):

“`disjunctive`” scheduling

Let $(x_1, \ldots, x_n)$ be a tuple of (integer/real)-valued variables indicating the starting time of a job j. Let $(p_1, \ldots, p_n)$ be the processing times of each job.

$$\texttt{disjunctive}([x_1, \ldots, x_n], [p_1, \ldots, p_n]) = \{[s_1, \ldots, s_n] \mid \forall i, j, i \neq j, \ (s_i + p_i \leq s_j) \vee (s_j + p_j \leq s_i)\}$$

In Gecode:

```
IntArgs p(4, 2,7,4,11);
unary(home, s, p);
```

Scheduling Constraints

In Resource Constrained Project Scheduling each resource can be used at most up to its capacity:

`cumulative` constraints [Aggoun and Beldiceanu, 1993]

- r_j release time of job j
- p_j processing time
- d_j deadline
- c_j resource consumption
- C limit not to be exceeded at any point in time

Let x be an n-tuple of (integer/real) value variables denoting the starting time of each job

$$\texttt{cumulative}([x_j],[p_j],[c_j],C) :=$$
$$\{([s_j],[p_j],[c_j],C) \mid \forall t \sum_{i \mid s_i \leq t \leq s_i+p_i} c_i \leq C\}$$

With $c_j = 1$ forall j and $C = 1 \leadsto$ `disjunctive`

Scheduling Constraints

`cumulatives` generalizes `cumulative` by: [Beldiceanu and Carlsson, 2002]

1. allowing to have several cumulative resurces and that each task has to be assigned to one of them
2. the resource consumption by any task is a variable that can take positive or negative values
3. it is possible to enforce the cumulated consumption to be less than or equal, or greater or equal to a given level.
4. the previous point on the cumulated resource consumption is enforced only for those time-points that are overalpped by at least 1 task. permitting multiple cumulative resources as well as negative resource consumptions by the tasks.

Scheduling Constraints
Cumulatives

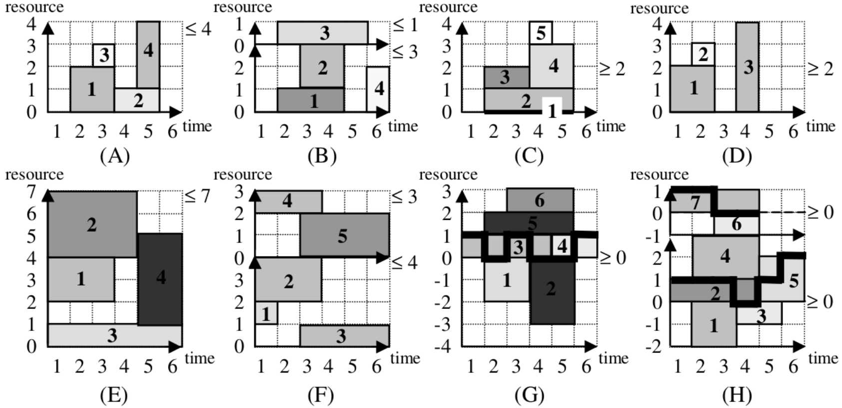
cumulatives constraints [Beldiceanu and Carlsson, 2002]

- variables $(y_j, x_j, d_j, c_j, e_j)$ for job $j \in J$
  $y_j \in \mathbb{Z}$ machine; $d_j \in \mathbb{Z}^+$ duration; $x_j \in \mathbb{Z}$ start time; $c_j \in \mathbb{Z}$ consumption; $e_j \in \mathbb{Z}$ end time
- parameters (r, L_r) for resource $r \in R$, L_r limit.
- constraint $\leq$ or $\geq$

$$\texttt{cumulatives}([y_j],[x_j],[d_j],[c_j],[e_j],[L_r],\lesseqgtr) :=$$

$$\Big\{([q_j],[s_j],[p_j],[u_j],[f_j],[L_{q_j}],\lesseqgtr) \mid$$
$$\forall j \in J: \; s_j + p_j = f_j \quad \text{and}$$
$$\forall j \in J, \forall t \in [s_j, e_j - 1], \hat{r} = y_j :$$
$$\sum_{\substack{i \mid s_i \leq t \leq s_i + p_i \\ y_i = y_j}} c_i \leq L_{\hat{r}} \Big\}$$

examples of cases modelled by `cumulatives`



from [Beldiceanu and Carlsson, 2002]

Others

- Sorted constraints (sorted(x, y))
- Bin-packing constraints (binpacking(l, b, s))
  l_j is the load variable of bin j, b_i the bin variable of item i, s_i size of item i
- Geometrical packing constraints (nooverlap)
  diffn$((x^1, \Delta x^1), \ldots, (x^m, \Delta x^m))$ arranges a given set of multidimensional boxes in n-space such that they do not overlap (aka, nooverlap)
- Value precedence constraints (precede(x, s, t))
- Logical implication: conditional$(\mathcal{D}, \mathcal{C})$ between sets of constrains $\mathcal{D} \Rightarrow \mathcal{C}$ (ite)

Examples

- Bin Packing

More (not in gecode)

- `clique`$(x|G,k)$ requires that a given graph contain a clique of size k
- `cycle`$(x|y)$ select edges such that they form exactly y directed cycles in a graph.
- `cutset`$(x|G,k)$ requires that for the set of selected vertices V', the set $V \setminus V'$ induces a subgraph of G that contains no cycles.

Global Constraint Catalog

Global Constraint Catalog

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NAME Keyword Meta-keyword Argument pattern Graph description
Bibliography Index

Keywords (ex: *Assignment, Bound consistency, Soft constraint,...*) can be searched by **Meta-keywords** (ex: *Application area, Filtering, Constraint type,...*)

About the catalogue

The catalogue presents a list of 348 global constraints issued from the literature in constraint programming and from popular constraint systems. The semantic of each constraint is given together with a description in terms of graph properties and/or automata.

CP Modeling Guidelines [Hooker, 2011]

1. A specially-structured subset of constraints should be replaced by a single global constraint that **captures the structure**, when a suitable one exists. This produces a more succinct model and can allow more effective filtering and propagation.

2. A global constraint should be replaced by a more specific one when possible, to exploit more effectively the special structure of the constraints.

3. The addition of redundant constraints (i..e, constraints that are implied by the other constraints) can improve propagation.

4. When two alternate formulations of a problem are available, including both (or parts of both) in the model may improve propagation. Different variables are linked through the use of channeling constraints.

References

Hooker J.N. (2011). **Hybrid modeling**. In *Hybrid Optimization*, edited by P.M. Pardalos, P. van Hentenryck, and M. Milano, vol. 45 of **Optimization and Its Applications**, pp. 11–62. Springer New York.

van Hoeve W. and Katriel I. (2006). **Global constraints**. In *Handbook of Constraint Programming*, chap. 6. Elsevier.