DM841
DISCRETE OPTIMIZATION

Part I

Constraint Propagation Algorithms

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Resume

- Definitions
  (CSP, restrictions, projections, istantiation, local consistency)
- Tigthtenings
- Global consistent (any instantiation local consistent can be extended to a solution) needs exponential time
  $\rightsquigarrow$ local consistency defined by condition Φ of the problem
- Tightenings by constraint propagation: reduction rules + rules iterations
  - reduction rules $\Leftrightarrow$ Φ consistency
  - rules iteration: reach fixed point, that is, closure of all tightenings that are Φ consistent

Outline

1. Local Consistency

2. Arc Consistency Algorithms

Node Consistency

We call a CSP node consistent if for every variable x every unary constraint on x coincides with the domain of x.

Example

- $\langle \mathcal{C}, x_1 \geq 0, \ldots, x_n \geq 0; x_1 \in \mathbb{N}, \ldots, x_n \in \mathbb{N} \rangle$
  and $\mathcal{C}$ does not contain other unary constraints
  node consistent

- $\langle \mathcal{C}, x_1 \geq 0, \ldots, x_n \geq 0; x_1 \in \mathbb{N}, \ldots, x_n \in \mathbb{Z} \rangle$
  and $\mathcal{C}$ does not contain other unary constraints
  not node consistent

A CSP is node consistent iff it is closed under the applications of the Node Consistency rule (propagator):

$$\frac{\langle C; x \in D \rangle}{\langle C; x \in C \cap D \rangle}$$

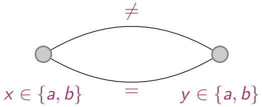
(the rule is parameterised by a variable x and a unary constraint C)

Arc Consistency

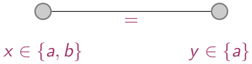
Arc consistency: every value in a domain is consistent with every binary constraint.

- $C = c(x, y)$ with $\mathcal{D} = \{D(x), D(y)\}$ is arc consistent iff
  - $\forall a \in D(x)$ there exists $b \in D(y)$ such that $(a, b) \in C$
  - $\forall b \in D(y)$ there exists $a \in D(x)$ such that $(a, b) \in C$
- $\mathcal{P}$ is arc consistent iff it is AC for all its binary constraints

In general arc consistency does not imply global consistency.
An arc consistent but inconsistent CSP:



A consistent but not arc consistent CSP:



Arc Consistency

A CSP is arc consistent iff it is closed under the applications of the Arc Consistency rules (propagators):

$$\frac{\langle C; x \in D(x), y \in D(y)\rangle}{\langle C; x \in D'(x), y \in D(y)\rangle}$$

where $D'(x) := \{a \in D(x) \mid \exists b \in D(y), (a, b) \in C\}$

$$\frac{\langle C; x \in D(x), y \in D(y)\rangle}{\langle C; x \in D(x), y \in D'(y)\rangle}$$

where $D'(y) := \{b \in D(y) \mid \exists a \in D(x), (a, b) \in C\}$

Exercise – Binary CSP

Theorem

An arbitrary (non-binary) CSP can be polynomially converted into an equivalent binary CSP.

Generalized Arc Consistency (GAC)

Given arbitrary (non-normalized, non-binary) $\mathcal{P}$, $C \in \mathcal{C}$, $x_i \in X(C)$

(Value) $v \in D(x_i)$ is consistent with C in $\mathcal{D}$ iff $\exists$ a valid tuple τ for C: $v_i = \tau[x_i]$. τ is called support for (x_i, v_i)

(Variable) $\mathcal{D}$ is GAC on C for x_i iff all values in $D(x_i)$ are consistent with C in $\mathcal{D}$ (i.e., $D(x_i) \subseteq \pi_{\{x_i\}}(C \cap \pi_{\{X(C)\}}(\mathcal{D}))$)

(Problem) $\mathcal{P}$ is GAC iff $\mathcal{D}$ is GAC for all x in X on all $C \in \mathcal{C}$

$\mathcal{P}$ is arc inconsistent iff the only domain tighter than $\mathcal{D}$ which is GAC for all variables on all constraints is the empty set.

(aka, hyperarc consistency, domain consistency)

Example

$\langle x = 1, y \in \{0,1\}, z \in \{0,1\}; \mathcal{C} = \{x \wedge y = z\}\rangle$
is hyperarc consistent

$\langle x \in \{0,1\}, y \in \{0,1\}, z = 1; \mathcal{C} = \{x \wedge y = z\}\rangle$
is not hyper-arc consistent

Example: arc consistency $\neq$ 2-consistency, AC $<$ 2C on non-normalized binary CSP, and incomparable on arbitrary CSP (later)

Generalized Arc Consistency

A CSP is arc consistent iff it is closed under the applications of the Arc Consistency rules (propagators):

$$\frac{\langle C; x_1 \in D(x), \ldots, x_k \in D(x_k)\rangle}{\langle C; x_1 \in D(x_1), \ldots, x_{i-1} \in D(x_{i-1}), x_i \in D'(x_i), x_{i+1} \in D(x_{i+1}), \ldots, x_k \in D(x_k)\rangle}$$

where $D'(x_i) := \{a \in D(x_i) | \exists \tau \in C, a = \tau[x_i]\}$

Outline

1. Local Consistency

2. Arc Consistency Algorithms

Arc Consistency

Arc Consistency rule 1 (propagator):

$$\frac{\langle C; x \in D(x), y \in D(y) \rangle}{\langle C; x \in D'(x), y \in D(y) \rangle}$$

where $D'(x) := \{a \in D(x) | \exists b \in D(y), (a, b) \in C\}$

This can also be written as ($\bowtie$ represents the join):

$$D(x) \leftarrow D(x) \cap \pi_{\{x\}}(\bowtie(C, D(y)))$$

Arc Consistency rule 2 (propagator):

$$\frac{\langle C; x \in D(x), y \in D(y) \rangle}{\langle C; x \in D(x), y \in D'(y) \rangle}$$

where $D'(y) := \{b \in D(y) | \exists a \in D(x), (a, b) \in C\}$

This can also be written as:

$$D(y) \leftarrow D(y) \cap \pi_{\{y\}}(\bowtie(C, D(x)))$$

Generalized Arc Consistency

(Generalized) Arc Consistency rule (propagator):

$$\frac{\langle C; x_1 \in D(x), \ldots, x_k \in D(x_k)\rangle}{\langle C; x_1 \in D(x_1), \ldots, x_{i-1} \in D(x_{i-1}), x_i \in D'(x_i), x_{i+1} \in D(x_{i+1}), \ldots, x_k \in D(x_k)\rangle}$$

where $D'(x_i) := \{a \in D(x_i) | \exists \tau \in C, a = \tau[x_i]\}$

This can also be written as:

$$D(x_i) \leftarrow D(x_i) \cap \pi_{\{x_i\}}(C \cap \pi_{X(C)}(\mathcal{D}))$$

AC1 – Reduction rule

Revision: making a constraint arc consistent by propagating the domain from a variable to anohter
Corresponds to:

$$D(x) \leftarrow D(x) \cap \pi_{\{x\}}(\bowtie(C, D(y)))$$

for a given variable x and constraint C
Assume normalized network:

REVISE$((x_i), x_j)$
input: a subnetwork defined by two variables $X = \{x_i, x_j\}$, a distinguished variable x_i, domains: D_i and D_j, and constraint R_{ij}
output: D_i, such that, x_i arc-consistent relative to x_j

```
for each a_i ∈ D_i
    if there is no a_j ∈ D_j such that (a_i, a_j) ∈ R_ij
        then delete a_i from D_i
    endif
endfor
```

Complexity: $O(d^2)$ or $O(rd^r)$
d values, r arity

AC1 – Rules Iteration

Binary case

AC-1($\mathcal{R}$)

input: a network of constraints $\mathcal{R} = (X, D, C)$
output: $\mathcal{R}'$ which is the loosest arc-consistent network equivalent to $\mathcal{R}$

repeat
    for every pair $\{x_i, x_j\}$ that participates in a constraint
        Revise$((x_i), x_j)$ (or $D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$)
        Revise$((x_j), x_i)$ (or $D_j \leftarrow D_j \cap \pi_j(R_{ij} \bowtie D_i)$)
    endfor
until no domain is changed

- Complexity (Mackworth and Freuder, 1986): $O(end^3)$
  e number of arcs, n variables
  (ed^2 each loop, a single succesful removal causes all loop again $\rightsquigarrow$ nd number of loops)
- best-case = $O(ed)$
- Arc-consistency is at least $O(ed^2)$ in the worst case (see later)
- $\rightsquigarrow$ too many calls to Revise

AC3 (Macworth, 1977)

General case – Arc oriented (coarse-grained)

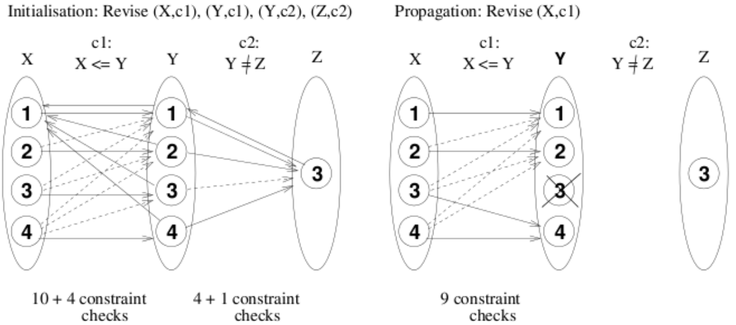
function `Revise3`(**in** x_i: variable; c: constraint): **Boolean** ;
begin
  CHANGE $\leftarrow$ **false**;
  foreach $v_i \in D(x_i)$ **do**
    if $\nexists \tau \in c \cap \pi_{X(c)}(D)$ *with* $\tau[x_i] = v_i$ **then**
      remove v_i from $D(x_i)$;
      CHANGE $\leftarrow$ **true**;
  return CHANGE ;
end

function `AC3/GAC3`(**in** X: set): **Boolean** ;
begin
  /* initalisation */;
  $Q \leftarrow \{(x_i, c) \mid c \in C, x_i \in X(c)\}$;
  /* propagation */;
  while $Q \neq \emptyset$ **do**
    select and remove (x_i, c) from Q;
    if `Revise`(x_i, c) **then**
      if $D(x_i) = \emptyset$ **then** return **false** ;
      else $Q \leftarrow Q \cup \{(x_j, c') \mid c' \in C \wedge c' \neq c \wedge x_i, x_j \in X(c') \wedge j \neq i\}$;
  return **true** ;
end

$O(er^3 d^{r+1})$ time
$O(er)$ space

AC3
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{D} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$



(a) (b)

AC4

Binary normalized problems – value oriented (fine grained)

function AC4(**in** X: set)**: Boolean** ;
begin
/* initialization */;
$Q \leftarrow \emptyset$; $S[x_j, v_j] = 0, \forall v_j \in D(x_j), \forall x_j \in X$;
foreach $x_i \in X, c_{ij} \in C, v_i \in D(x_i)$ **do**
initialize counter$[x_i, v_i, x_j]$ to $|\{v_j \in D(x_j) \mid (v_i, v_j) \in c_{ij}\}|$;
if *counter*$[x_i, v_i, x_j] = 0$ **then** remove v_i from $D(x_i)$ and add (x_i, v_i) to Q;
add (x_i, v_i) to each $S[x_j, v_j]$ s.t. $(v_i, v_j) \in c_{ij}$;
if $D(x_i) = \emptyset$ **then** return **false** ;
/* propagation */;
while $Q \neq \emptyset$ **do**
select and remove (x_j, v_j) from Q;
foreach $(x_i, v_i) \in S[x_j, v_j]$ **do**
if $v_i \in D(x_i)$ **then**
counter$[x_i, v_i, x_j]$ = counter$[x_i, v_i, x_j] - 1$;
if *counter*$[x_i, v_i, x_j] = 0$ **then**
remove v_i from $D(x_i)$; add (x_i, v_i) to Q;
if $D(x_i) = \emptyset$ **then** return **false** ;
return **true** ;
end

$O(ed^2)$ time (optimal)
$O(ed^2)$ space
$O(erd^r)$ time for GAC

AC4
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$

$$\begin{array}{lll}
\texttt{counter}[x, 1, y] = 4 & \texttt{counter}[y, 1, x] = 1 & \texttt{counter}[y, 1, z] = 1 \\
\texttt{counter}[x, 2, y] = 3 & \texttt{counter}[y, 2, x] = 2 & \texttt{counter}[y, 2, z] = 1 \\
\texttt{counter}[x, 3, y] = 2 & \texttt{counter}[y, 3, x] = 3 & \texttt{counter}[y, 3, z] = 0 \\
\texttt{counter}[x, 4, y] = 1 & \texttt{counter}[y, 4, x] = 4 & \texttt{counter}[y, 4, z] = 1 \\
 & & \texttt{counter}[z, 3, y] = 3
\end{array}$$

$$\begin{array}{ll}
S[x, 1] = \{(y, 1), (y, 2), (y, 3), (y, 4)\} & S[y, 1] = \{(x, 1), (z, 3)\} \\
S[x, 2] = \{(y, 2), (y, 3), (y, 4)\} & S[y, 2] = \{(x, 1), (x, 2), (z, 3)\} \\
S[x, 3] = \{(y, 3), (y, 4)\} & S[y, 3] = \{(x, 1), (x, 2), (x, 3)\} \\
S[x, 4] = \{(y, 4)\} & S[y, 4] = \{(x, 1), (x, 2), (x, 3), (x, 4), (z, 3)\} \\
 & S[z, 3] = \{(y, 1), (y, 2), (y, 4)\}
\end{array}$$

AC6

Binary normalized problems

$S[x_j, v_j]$ list of values (x_i, v_i) currently having (x_j, v_j) as their first support

function AC6(**in** X: set): **Boolean** ;
  begin
    /* initialization */;
    $Q \leftarrow \emptyset$; $S[x_j, v_j] = 0, \forall v_j \in D(x_j), \forall x_j \in X$;
    foreach $x_i \in X, c_{ij} \in C, v_i \in D(x_i)$ **do**
        $v_j \leftarrow$ smallest value in $D(x_j)$ s.t. $(v_i, v_j) \in c_{ij}$;
        if v_j *exists* **then** add (x_i, v_i) to $S[x_j, v_j]$;
        else remove v_i from $D(x_i)$ and add (x_i, v_i) to Q;
        if $D(x_i) = \emptyset$ **then** return **false** ;
    /* propagation */;
    while $Q \neq \emptyset$ **do**
        select and remove (x_j, v_j) from Q;
        foreach $(x_i, v_i) \in S[x_j, v_j]$ **do**
            if $v_i \in D(x_i)$ **then**
                $v'_j \leftarrow$ smallest value in $D(x_j)$ greater than v_j s.t. $(v_i, v_j) \in c_{ij}$;
                if v'_j *exists* **then** add (x_i, v_i) to $S[x_j, v'_j]$;
                else
                    remove v_i from $D(x_i)$; add (x_i, v_i) to Q;
                    if $D(x_i) = \emptyset$ **then** return **false** ;
    return **true** ;
  end

$O(ed^2)$ time
$O(ed)$ space

AC6
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\},$$
$$\mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$

$$\begin{aligned} S[x,1] &= \{(y,1),(y,2),(y,3),(y,4)\} \\ S[x,2] &= \{\} \\ S[x,3] &= \{\} \\ S[x,4] &= \{\} \end{aligned}$$

$$\begin{aligned} S[y,1] &= \{(x,1),(z,3)\} \\ S[y,2] &= \{(x,2)\} \\ S[y,3] &= \{(x,3)\} \\ S[y,4] &= \{(x,4)\} \\ S[z,3] &= \{(y,1),(y,2),(y,4)\} \end{aligned}$$

Reverse2001

Binary case

function Revise2001(**in** x_i: variable; c_{ij}: constraint)**: Boolean** ;
begin
CHANGE ← **false**;
foreach $v_i \in D(x_i)$ *s.t.* *Last*$(x_i, v_i, x_j) \notin D(x_j)$ **do**
$v_j \leftarrow$ smallest value in $D(x_j)$ greater than Last(x_i, v_i, x_j) s.t. $(v_i, v_j) \in c_{ij}$;
if v_j *exists* **then** Last$(x_i, v_i, x_j) \leftarrow v_j$;
else
remove v_i from $D(x_i)$;
CHANGE ← **true**;
return CHANGE ;
end

function AC3/GAC3(**in** X: set)**: Boolean** ;
begin
/* initalisation */;
$Q \leftarrow \{(x_i, c) \mid c \in C, x_i \in X(c)\}$;
/* propagation */;
while $Q \neq \emptyset$ **do**
select and remove (x_i, c) from Q;
if *Revise*(x_i, c) **then**
if $D(x_i) = \emptyset$ **then** return **false** ;
else $Q \leftarrow Q \cup \{(x_j, c') \mid c' \in C \wedge c' \neq c \wedge x_i, x_j \in X(c') \wedge j \neq i\}$;
return **true** ;
end

$O(ed^2)$ time
$O(ed)$ space

Reverse2001
Example

$$\mathcal{P} = \langle X = (x, y, z),\ \mathcal{DE} = \{D(x) = D(y) = \{1, 2, 3, 4\}, D(z) = \{3\}\}, \\ \mathcal{C} = \{C_1 \equiv x \leq y, C_2 \equiv y \neq z\}\}\rangle$$

$\texttt{Last}[x, 1, y] = 1$	$\texttt{Last}[y, 1, x] = 1$	$\texttt{Last}[y, 1, z] = 3$
$\texttt{Last}[x, 2, y] = 2$	$\texttt{Last}[y, 2, x] = 1$	$\texttt{Last}[y, 2, z] = 3$
$\texttt{Last}[x, 3, y] = 3$	$\texttt{Last}[y, 3, x] = 1$	$\texttt{Last}[y, 3, z] = nil$
$\texttt{Last}[x, 4, y] = 4$	$\texttt{Last}[y, 4, x] = 1$	$\texttt{Last}[y, 4, z] = 3$
		$\texttt{Last}[z, 3, y] = 1$

Limitation of Arc Consistency

Example

$$\langle x < y, y < z, z < x; x, y, z \in \{1..100000\}\rangle$$

is inconsistent.

Proof: Apply revise to $(x, x < y)$

$$\langle x < y, y < z, z < x; x \in \{1..99999\}, y, z \in \{1..100000\}\rangle,$$

ecc. we end in a fail.

- Disadvantage: large number of steps.
  Run time depends on the size of the domains!
- Note: we could prove fail by transitivity of $<$.
  $\rightsquigarrow$ Path consitency involves two constraints together