

DM841  
DISCRETE OPTIMIZATION

Part I  
**Propagation Events and Implementations**

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1. Generic Rules Iteration

2. Systems

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2. Systems

## Algorithms for constraint propagation:

- ▶ scheduling steps of atomic reduction
- ▶ termination criterion: local consistency
  
- ▶ How to schedule the application of reduction rules to guarantee termination?
- ▶ How to avoid (at low cost) the application of redundant rules?
- ▶ Have all derivations the same result?
- ▶ How can we characterize it?

# Propagators

- ▶ Given  $\mathcal{P}$  a **reduction rule** is a function  $f$  from  $\mathcal{S}_{\mathcal{P}}$  to  $\mathcal{S}_{\mathcal{P}}$  for all  $\mathcal{P}' \in \mathcal{S}_{\mathcal{P}}$ ,  $f(\mathcal{P}') \in \mathcal{S}_{\mathcal{P}'}$ .  
(most cases take care of one a single variable and a single constraint)
- ▶ Given  $\mathcal{P}$  a **propagator**  $f$  for  $C$  is a reduction rule from  $\mathcal{S}_{\mathcal{P}}$  to  $\mathcal{S}_{\mathcal{P}}$  that tightens only domains independently of the constraints other than  $C$ .
- ▶ A propagator  $f$  is **correct** for  $C$  iff it does not remove any assignment for  $C$ :  $\{a \in D\} \cap C = \{a \in f(D)\} \cap C$

Systems consider set of propagators to implement a constraint  
(However global constraints have a single propagator.)

### Example

$$C \equiv x_1 \leq x_2 + 1$$

$$f(D, x_1) = p(D)(x_1) = \{n \in D(X_1) \mid n \leq \max_D\{x_2\} + 1\}$$

$$\text{input}(p) = x_2, \text{output}(p) = x_1$$

# Propagators

Properties of propagators:

- ▶ A propagator  $f$  is:
  - ▶ **contracting** (or decreasing): for all  $\mathcal{P} \in \mathcal{S}_{\mathcal{P}}$ :  $f(\mathcal{P}) \leq \mathcal{P}$ , that is:  
 $\mathcal{D}(f(\mathcal{P})) \subseteq \mathcal{D}(\mathcal{P})$
- ▶ A propagator  $f$  can be:
  - ▶ **monotonic** if  $\mathcal{P}_1 \leq \mathcal{P}_2 \Rightarrow f(\mathcal{P}_1) \leq f(\mathcal{P}_2)$
  - ▶ **commuting** if  $fg(\mathcal{P}) = gf(\mathcal{P})$
  - ▶ **idempotent** for  $\mathcal{P}$  if  $f(f(\mathcal{P})) = f(\mathcal{P})$  (weak: for some  $\mathcal{P} \in \mathcal{S}_{\mathcal{P}}$ )
  - ▶ **subsumed** (or entailed) by  $\mathcal{P}$  iff  $\forall \mathcal{P}_1 \leq \mathcal{P} : f(\mathcal{P}_1) = \mathcal{P}_1$  (strong: for all  $\mathcal{P} \in \mathcal{S}_{\mathcal{P}}$ )

Eg:

$$f(D, x) = D(x) \cap \{1, 2, 3\}$$

implementing the domain constraint  $x \in \{1, 2, 3\}$ . After  $f$  has been executed once, there is no point to execute  $f$  again as for all  $D'$

$$D' \leq f(D) \implies f(D') = D'$$

(particular case when all variables are instantiated)

## Example

$$\mathcal{P}_1 = \langle X = (x_1, x_2); D_1(x_1) = \{1, 2\}, D_1(x_2) = \{2\}; \mathcal{C} \equiv \{x_1 = x_2\} \rangle$$

$$\mathcal{P}_2 = \langle X = (x_1, x_2); D_2(x_1) = \{1, 2, 3\}, D_2(x_2) = \{2\}; \mathcal{C} \equiv \{x_1 = x_2\} \rangle$$

$f$  removes values from  $D(x_1)$  that have no support on  $\mathcal{C}$  if less than half of them have support.

- ▶  $f(D_2(x_1)) = \{2\}$
- ▶  $D(f(\mathcal{P}_1)) \not\subseteq D(f(\mathcal{P}_2))$  whereas  $D(\mathcal{P}_1) \subseteq D(\mathcal{P}_2) \rightsquigarrow$  not monotonic

$g$  removes one of the values from  $x_1$  that have no support on  $\mathcal{C}$  if such a value exists.

- ▶  $g(D_2(x_1)) = \{1, 2\}, gg(D_2(x_1)) = \{2\}$
- ▶  $gg(\mathcal{P}_1) \neq g(\mathcal{P}_2)$



- ▶ **Iteration:** Let  $\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \rangle$  and  $F = \{f_1, \dots, f_k\}$  a finite set of propagators on  $\mathcal{S}_{\mathcal{P}}$ . An iteration of  $F$  on  $\mathcal{P}$  is a sequence  $\langle \mathcal{P}_0, \mathcal{P}_1, \dots \rangle$  of elements of  $\mathcal{S}_{\mathcal{P}}$  defined by

$$\begin{aligned} \mathcal{P}_0 &= \mathcal{P} \\ &\vdots \\ \mathcal{P}_j &= f_{n_j}(\mathcal{P}_{j-1}) \end{aligned}$$

where  $j > 0$  and  $n_j \in [1, \dots, k]$ .

- ▶  $\mathcal{P}$  is **stable** for  $F$  iff  $\forall f \in F, f(\mathcal{P}) = \mathcal{P}$
- ▶ There may be several stable  $\mathcal{P}$  but if  $F$  are monotonic then **unique**
- ▶ Let  $\mathcal{P} = \langle X, \mathcal{D}, \mathcal{C} \rangle$  and  $F = \{f_1, \dots, f_k\}$ . If  $\langle \mathcal{P}_0, \mathcal{P}_1, \dots \rangle$  is infinite iteration of  $F$  where each  $f \in F$  is activated infinitely often then there exists  $j \geq 0$  such that  $\mathcal{P}_j$  is stable for  $F$  ( $\equiv j$  is finite!)
- ▶ If  $\mathcal{P}$  is stable for  $F$  then it is its **weakest simultaneous fixed point** (greatest mutual fixed point of all propagators).  
A strongest simultaneous fixed point would be a solution (hence possibly not unique) which would not violate solution preservation

# Iteration of Reduction Rules

```
procedure Generic-Iteration( $N, F$ );  
   $G \leftarrow F$ ;  
  while  $G \neq \emptyset$  do  
    select and remove  $g$  from  $G$ ;  
    if  $N \neq g(N)$  then  
      update( $G$ );  
       $N \leftarrow g(N)$ ;  
  
  /* update( $G$ ) adds to  $G$  at least all functions  $f$  in  $F \setminus G$  for which  
   $g(N) \neq f(g(N))$  */
```

If the propagator is contracting then Generic-Iteration terminates.  
If propagator is monotonic then the final result does not change with the order in which propagators are applied.

If propagators in addition to monotonic are also idempotent and commutative then:

```
procedure Direct-Iteration( $N, F$ );  
   $G \leftarrow F$ ;  
  while  $G \neq \emptyset$  do  
    select and remove  $g$  from  $G$ ;  
     $N \leftarrow g(N)$ ;
```

# Iteration of Reduction Rules

## Example

Recall for arc consistency:

Arc Consistency rule 1 (propagator):

$$\frac{\langle C; x \in D(x), y \in D(y) \rangle}{\langle C; x \in D'(x), y \in D(y) \rangle}$$

where  $D'(x) := \{a \in D(x) \mid \exists b \in D(y), (a, b) \in C\}$

Set of propagators  $F_{AC} = \{f_{ij} \mid x_i \in X, c_j \in C\}$  all monotonic.  $\Rightarrow$  terminates in arc consistency closure, which is fixed point for  $F_{AC}$ .

# Improvements

Generic iteration is an example of [propagator engine](#)

What makes it naive?

- ▶ Termination relies on strict contraction
- ▶ We always have to check all propagators for one that can strictly contract

Ideas:

- ▶ Maintain propagators which are known to be at fixpoint
- ▶ Look at the variables that propagators actually compute with  
Dependency-directed propagation

Fixpoint knowledge avoids useless execution (idempotence, subsumption)  
knowledge provided by propagator

Generic iteration is an example of **propagator engine**

```
propagate( $P_f, P_n, D$ )
1:  $N \leftarrow P_n$ 
2:  $P \leftarrow P_f \cup P_n$ 
3: while  $N \neq \emptyset$  do
4:    $p \leftarrow \text{select}(N)$ 
5:    $N \leftarrow N - \{p\}$ 
6:    $D' \leftarrow p(D)$ 
7:    $M \leftarrow \{x \in \mathcal{V} \mid D(x) \neq D'(x)\}$ 
8:    $N \leftarrow N \cup \{p' \in P \mid \text{input}(p') \cap M \neq \emptyset\}$ 
11:   $D \leftarrow D'$ 
12: return  $D$ 
```

$P_f$  is set of propagators at fixed point (idempotent or subsumed)

**Scheduling**  $p$ : adding a propagator to the set  $N$  (not known to be at fixed point). Yet undefined how a propagator is chosen from  $N$

Note: search can be seen as doing incremental propagation

# Improvements: Events

Most solvers implement arithmetic-oriented propagators

↪ a reduction of a domain of a variable has different implications depending on the type of reduction

Four types of **Events**:

- ▶ Any or RemValue: when a value  $v$  is removed from  $D(x_i)$
- ▶ Min or IncMin: when the minimum value of  $D(x_i)$  increases
- ▶ Max or DecMax: when the maximum value of  $D(x_i)$  decreases
- ▶ Fix or Instantiate: when  $D(x_i)$  becomes a singleton

## Modified AC3 to handle parameter Mtype (modification type)

```

function Constraint-Propag(in  $X$ : set): Boolean ;
  begin
1    foreach  $c \in C$  do perform init-propag on  $c$  and update  $Q$  with relevant
      events;
2    while  $Q \neq \emptyset$  do
3      select and remove  $(x_i, c, x_j, Mtype)$  from  $Q$ ;
4      if  $Revise(x_i, c, (x_j, Mtype), Changes)$  then
5        if  $D(x_i) = \emptyset$  then return false ;
6        foreach  $c' \in \Gamma^C(x_i), Mtype \in Changes$  do
7          foreach  $x_j \in X(c'), j \neq i$  do  $Q \leftarrow Q \cup \{(x_j, c', x_i, Mtype)\}$ ;
8    return true ;
  end
  /*  $\Gamma^C(x_i)$  is the set of constraints with  $x_i$  in their scheme */

```

The presence of  $(x_j, c, x_i, Mtype)$  in  $Q$  means that  $x_j$  should be revised on  $c$  because of an  $Mtype$  change in  $D(x_i)$ .

Process constraint propagation differently according to the type of event

```

function revise(inout  $x_i$ ; in  $c \equiv x_{k_1} \leq x_{k_2}$ ; in  $(x_j, Mtype)$ ; out  $Changes$ ):
  Boolean ;
   $Changes \leftarrow \emptyset$ ;
  switch  $Mtype$  do
    case  $RemValue$ 
      nothing;
    case  $IncMin$ 
      if  $j = k_1$  then remove all  $v < min_D(x_j)$  from  $D(x_i)$ ;
    case  $DecMax$ 
      if  $j = k_2$  then remove all  $v > max_D(x_j)$  from  $D(x_i)$ ;
    case  $Instantiate$ 
      if  $j = k_1$  then remove all  $v < min_D(x_j)$  from  $D(x_i)$ ;
      else remove all  $v > max_D(x_j)$  from  $D(x_i)$ ;
   $Changes \leftarrow$  the types of changes performed on  $D(x_i)$ ;
  
```

Also: for a certain constraint it can be that a given event cannot alter the other variables of the constraint. Hence it makes sense to:

6: **foreach**  $c' \in \Gamma_{Mtype}^c(x_j), Mtype \in Changes$  **do** ...

Example. Let  $c \equiv x_1 \leq x_2$ . The only events that require propagation are IncMin and Instantiate on  $x_1$ , and DecMax and Instantiate on  $x_2$ .



```

3   select and remove  $(x_i, c, x_j, Mtype, \Delta_j)$  from  $Q$ ;
4   if  $\text{Revise}(x_i, c, (x_j, Mtype, \Delta_j), \text{Changes}, \Delta_i)$  then
5       if  $D(x_i) = \emptyset$  then return false;
6       foreach  $c' \in \Gamma_{Mtype}^C(x_i), Mtype \in \text{Changes}$  do
7           foreach  $x_j \in X(c'), j \neq i$  do  $Q \leftarrow Q \cup \{(x_j, c', x_i, Mtype, \Delta_i)\}$ 

```

---

```

function revise(inout  $x_i$ ; in  $c \equiv x_{k_1} = x_{k_2} + m$ ; in  $(x_j, Mtype, \Delta_j)$ ;
                out  $\text{Changes}$ ; out  $\Delta_i$ ): Boolean ;

```

```

     $\text{Changes} \leftarrow \emptyset$ ;

```

```

    switch  $Mtype$  do

```

```

        case  $RemValue$ 

```

```

            if  $j = k_1$  then foreach  $v \in \Delta_j$  do remove  $(v - m)$  from  $D(x_i)$ ;

```

```

            else foreach  $v \in \Delta_j$  do remove  $(v + m)$  from  $D(x_i)$ ;

```

```

        case  $IncMin$ 

```

```

            if  $j = k_1$  then remove all  $v < \min_D(x_j) - m$  from  $D(x_i)$ ;

```

```

            else remove all  $v < \min_D(x_j) + m$  from  $D(x_i)$ ;

```

```

        case  $DecMax$ 

```

```

            if  $j = k_1$  then remove all  $v > \max_D(x_j) - m$  from  $D(x_i)$ ;

```

```

            else remove all  $v > \max_D(x_j) + m$  from  $D(x_i)$ ;

```

```

        case  $Instantiate$ 

```

```

            if  $j = k_1$  then assign  $\min_D(x_j) - m$  to  $x_i$ ;

```

```

            else assign  $\min_D(x_j) + m$  to  $x_i$ ;

```

```

     $\text{Changes} \leftarrow$  the types of changes performed;

```

```

     $\Delta_i \leftarrow$  all values removed from  $D(x_i)$ ;

```

# More Optimization

## Priorities

### Choose propagator

- ▶ according to cost: cheapest first
- ▶ according to expected impact
- ▶ general (queue): last-in last-out (starvation avoided), first-in first-out

Another observation:  
propagator for

$$\max(x, y) = z$$

and values for  $x$  are smaller than for  $y$   
Replace by propagator for  $y = z$

1. Generic Rules Iteration

2. Systems

- ▶ Detecting failure and entailment
- ▶ Domains: single data structure continuously updated.  
constraint store  $\equiv$  domain extension  $\mathcal{D}$
- ▶ State restoration
- ▶ Finding dependent propagators (compute events and find propagators)
- ▶ Variables for propagators

- ▶ Events
- ▶ Selecting next propagator

# Variable Domains

- ▶ Domain representation  
range sequence:  $s = \{[n_1, m_1], \dots, [n_k, m_k]\}$  (singly/doubly linked lists)  
bit vector
- ▶ Value operations  
`x.getmin()`, `x.getmax()`, `x.hasval()`, `x.adjmin(n)`,  
`x.adjmax(n)`, `x.excval(n)`
- ▶ Iterators:

```
for (IntVarValues i(x); i(); ++i)
    std::cout << i.val() << ' ';

for (IntVarRanges i(x); i(); ++i)
    std::cout << i.min() << ".." << i.max() << ' ';
```

- ▶ Domain operations
- ▶ Subscriptions ( $p$  is executed whenever the domains of one of its variables changes according to an event). Options:
  - ▶ list  $E_i, p_i$  pair event propagator that requires execution
  - ▶ a list for each event and one for each propagator
  - ▶ array of propagators partitioned by events

<b>Operations</b>	<b>Range sequence</b>	<b>Bitvector</b>
$x.getmin()$	$O(1)$	$O(1)$
$x.getmax()$	$O(1)$	$O(1)$
$x.hasval(n)$	$O(r)$	$O(1)$
$x.adjmin(n)$	$O(r)$	$O(1)$
$x.adjmax(n)$	$O(r)$	$O(1)$
$x.excv(n)$	$O(r)$	$O(v)$
$i.done()$	$O(1)$	$O(v)$
$i.value()$	$O(1)$	$O(1)$
$i.next()$	$O(1)$	$O(v)$

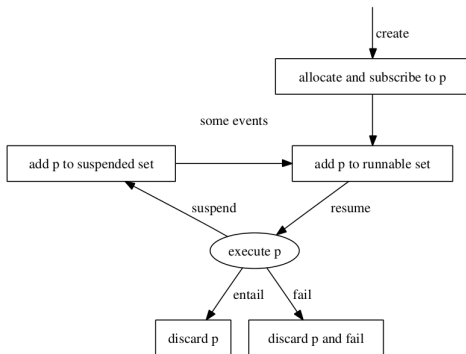


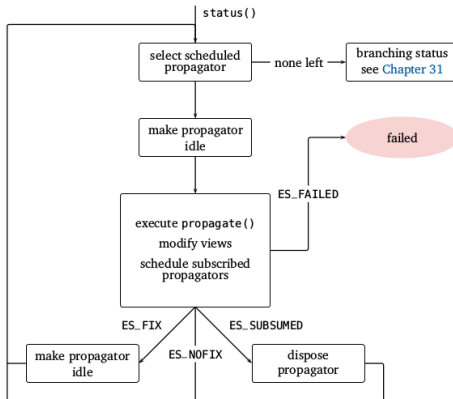
# Propagators

Piece of software with some private state that implements a constraint  $C$  over some variables or *parameters*

The algorithm implemented is called **filtering algorithm**. It uses value and domain operations and raises events that cause scheduling of other propagators

Life cycle





- ▶ Idempotency: it may be costly and difficult to guarantee. Some propagators return a state:
  - ▶ fixpoint (weak idempotent, ie, with respect to  $\times$  rather than for all),
  - ▶ no fixpoint (we do not know),
  - ▶ subsumed (entailed),
  - ▶ failure.

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