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DM559/DM545 – Linear and integer programming

Sheet 1, Spring 2018 [pdf format]

This exercise sheet is about modeling optimization problems in linear programming terms.
Recall that you have to identify and denote mathematically the:

i) parameters

ii) variables

and express as a linear combination of those terms the

iii) objective function

iv) constraints.

The description of the model has to be organized in the following parts:

1. **Notation.** Introduction of the mathematical notation: which symbols denote parameters and variables? which indices are you using and where are they running?
2. **Model.** Mathematical model:

$$\begin{aligned} \max \quad & \mathbf{c}^t \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

   (Most likely you will express the model in scalar notation and summations. Remember not to leave any index unspecified. Watch out the quantifiers.)
3. **Explanation.** Explanation of each line of the mathematical model.

Remember to check that every symbol and index in your model is defined.
See Sec. 1.3.2 from the Lecture Notes [LN] for an example of the expected modeling process.

Note that at the exam your answers must be digitalized, hence it is good to start becoming acquainted with different tools to produce text documents containing mathematical notation and graphs. Read the "Instructions for Written Exam" in the Assessment section of the course web page for a list of useful tools.

Beside these exercises, it is really important that you read Chapter 2 "Examples" from the book [MG]. You find this book downloadable from SDU network. Alternatively you can try from home after establishing a VPN connection: https://www.sdu.dk/en/om_sdu/faellesomraadet/it-service/services/netvaerksadgang/vpn.

Exercise 1* Food manufacture

A chocolate factory produces two types of chocolate bars: Milk & Hazelnuts (M&H) chocholate and Dark (D) chocholate. The price for a pack of M&H and a pack of D is 100 and 160 Dkk, respectively. Each pack of M&H is made of 50 grams of hazelnuts, 60 grams of chocolate, 40 grams of milk and 50 grams of sugar. Each Dark bar contains 150 grams of chocolate, 50 grams of fat and 30 grams of sugar. The factory has 200 grams of hazelnuts, 1000 grams of chocolate, 250 grams of milk, 300 grams of sugar, and 300 grams of fat left.
Your goal is to determine how many M&H bars and D bars the company should produce to maximize its profit.

a) Give the linear program for the problem, using variables x_1 and x_2 and the parameters defined above. Specify (i) the constraints, and (ii) the objective function.

b) Graph the feasible region of your linear program. (Tip: look at the "Tools" section in the course web page and find an application that can help you in this task. Those using MacOsx have also the option of the program Grapher).

c) Compute the profit at each vertex of the feasible region, and report the best solution.

Exercise 2* Optimal Blending

The Metalco Company wants to blend a new alloy (metal) made by 40 percent tin, 35 percent zinc, and 25 percent lead from 5 available alloys having the following properties:

	Alloy				
Property	1	2	3	4	5
Percentage of tin	60	25	45	20	50
Percentage of zinc	10	15	45	50	40
Percentage of lead	30	60	10	30	10
Cost (DKK/Kg)	77	70	88	84	94

The objective is to determine the proportions of these alloys that should be blended to produce the new alloy at a minimum cost. Formulate a linear programming model for this problem. [Problem from ref. HL]

To help you in the task we start introducing the mathematical notation that will be used for the model. Let $J = \{1, 2, \dots, 5\}$ indexed by j be the set of alloys and $I = \{\text{tin}, \text{zinc}, \text{lead}\}$ indexed by i be the set of metals. Let a_{ij} be the fixed parameters that determine the percentage amount of metal i in alloy j. Let c_j be the cost of alloy j in Dkk per Kg. The problem asks to determine the proportions of the alloys to blend to obtain the new alloy with the properties of 40% tin, 35% zinc and 25% lead. Let's call these last parameters $b_i, i \in I$ and the proportion of each alloy to blend with respect to the new alloy by $y_j \geq 0$. Specify the constraints and the objective function using the mathematical terms introduced.
[Units of measure offer a way to test the correctness of the model. In particular, the proportion y_j can be seen as the amount of alloy in Kg per amount of new alloy. Let α be the amount in Kg of the new alloy, and let x_j be the amount in Kg of each alloy from I. Then $y_j = x_j/\alpha$ and it is an adimensional quantity.]

Exercise 3*

A cargo plane has three compartments for storing cargo: front, center and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	Weight Capacity (Tons)	Space Capacity (Cubic meters)
Front	12	7000
Center	18	9000
Back	10	5000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.
The following four cargos have been offered for shipment on an upcoming flight as space is available:

Cargo	Weight(Tons)	Volume (Cubic meters/Tons)	Profit
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Formulate a linear programming model for this problem. [Problem from ref. HL]

Exercise 4*

A small airline flies between three cities: Copenhagen, Aarhus, and Odense. They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Copenhagen, stops in Odense, and continues to Aarhus. There are three types of passengers:

- (a) Those traveling from Copenhagen to Odense.
- (b) Those traveling from Odense to Aarhus.
- (c) Those traveling from Copenhagen to Aarhus.

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

- (a) Y class: full coach.
- (b) B class: nonrefundable.
- (c) M class: nonrefundable, 3-week advanced purchase.

Ticket prices, which are largely determined by external influences (i.e., train and bus competitors), have been set and advertised as follows:

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	300	160	360
B	220	130	280
M	100	80	140

Based on past experience, demand forecasters at the airline have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	4	8	3
B	8	13	10
M	22	20	18

The goal is to decide how many tickets from each of the 9 origin/destination/ fare-class combinations to sell. The constraints are that the plane cannot be overbooked on either of the two legs of the flight and that the number of tickets made available cannot exceed the forecasted maximum demand. The objective is to maximize the revenue.
Formulate this problem as a linear programming problem.

Exercise 5*

In this exercise we study the application of linear programming to an area of statistics, namely, regression.

Consider a set of $m = 9$ measurements: $28, 62, 80, 84, 86, 86, 92, 95, 98$. A way to summarize these data is by their mean

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$$

An alternative way is by the median, ie, the measurement that is worse than half of the other scores and better than the other half.
There is a close connection between these statistics and optimization. Show that the mean is the measure that minimizes the sum of squared deviation between the data points and itself and that the median minimizes the sum of the absolute values of the differences between each data point and itself.

Consider now a set of points on a two-dimensional space $S = (x_1, y_1), \dots, (x_m, y_m)$. The points are measurements of a response variable given some control variable, for example, blood pressure given the weight of a person. The points hint at a linear dependency between the variables representing the two dimensions. We may assume a random fluctuation around the right value and hence the following regression model:

$$y = ax + b + \epsilon$$

Specifically, for our set of points S we have

$$y_i = ax_i + b + \epsilon_i, \quad i = 1, \dots, m$$

We thus want to find a line that best fits the measured points. That is, we wish to determine the (unknown) numbers a and b. There is no unique criterion to formulate the desire that a given line "best fits" the points. The task can be achieved by minimizing, in some sense, the vector ϵ. As for the mean and median, we can consider minimizing either the sum of the squares of the ϵ_i's or the sum of the absolute values of the ϵ_i's. These concepts are formalized in measure theory by the so called L^p-norm ($1 \leq p < \infty$). For the vector ϵ:

$$\epsilon_{\ p} = \left(\sum_i \epsilon_i^p \right)^{1/p}$$

The method of least squares, which is perhaps the most popular, corresponds to L^2-norm. The minimization of L^2-norm for the vector ϵ in the variables a and b has a closed form solution that you may have encountered in the statistics courses. This method needs not always to be the most suitable, however. For instance, if a few exceptional points are measured with very large error, they can influence the resulting line a great deal. Just as the median gives a more robust estimate of a collection of numbers than the means, the L^1 norm is less sensitive to outliers than least square regression is. The problem is to solve the following minimization problem:

$$\text{argmin}_{a,b} \sum_i |\epsilon_i| = \text{argmin}_{a,b} \sum_i |ax_i + b - y_i|$$

Unlike for least square regression, there is no explicit formula for the solution of the L^1-regression problem. However the problem can be formulated as a linear programming problem. Show how this can be done.

The regression via L^∞-norm corresponds to solving the problem:

$$\text{argmin} \max_{i=1}^{n} |ax_i + b - y_i|$$

This problem can also be solved by linear programming. Show how to formulate the problem as a linear program.

Exercise 6 Dynamic Input-Output Model

This problem is the dynamic version of Leontief's Input-Output model. A treatment of the model from the point of view of LP is available in Sec. 5.2 of [Wi] (the book is available as ebook from SDU).
An economy consists of three industries: coal, steel and transport. Each unit produced by one of the industries (a unit will be taken as DKK 1's worth of value of production) requires inputs from possibly its own industry as well as other industries. The required inputs and the manpower requirements (also measured in DKK) are given in the Table 1. There is a time lag in the economy so that output in year $t + 1$ requires an input in year t.
Output from an industry may also be used to build productive capacity for itself or other industries in future years. The inputs required to give unit increases (capacity for DKK 1's worth of extra production) in productive capacity are given in Table 2. Input from an industry in year t results in a (permanent) increase in productive capacity in year $t + 2$.

	Outputs (year $t+1$), production		
Inputs (year t)	Coal	Steel	Transport
Coal	0.1	0.5	0.4
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.6	0.3	0.2

Table 1:

	Outputs (year $t+2$), productive capacity		
Inputs (year t)	Coal	Steel	Transport
Coal	0.0	0.7	0.9
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.4	0.2	0.1

Table 2:

Stocks of goods may be held from year to year. At present (year 0) the stocks and productive capacities (per year) are given in Table 3 (in DKKm) There is a limited yearly manpower capacity of DKK 470m. It is wished to investigate different possible growth patterns for the economy over the next five years. In particular it is desirable to know the growth patterns which would result from pursuing the following objectives:

(i) Maximiging total productive capacity at the end of the five years while meeting an exogenous consumption requirement of DKK 60m of coal DKK 60m of steel and DKK 30m of transport in every year (apart from year 0).

(ii) Maximizing total production (rather than productive capacity) in the fourth and fifth years, but ignoring exogenous demand in each year.

(iii) Maximizing the total manpower requirement (ignoring the manpower capacity limitation) over the period while meeting the yearly exogenous demands of (i).

	Year 0	
	Stocks	Productive capacity
Coal	150	300
Steel	80	350
Transport	100	280

Table 3: