Department of Mathematics and Computer Science
University of Southern Denmark, Odense

April 5, 2018
Marco Chiarandini

DM559/DM545 – Linear and integer programming

Sheet 3, Spring 2018 [pdf format]

Exercise 1* Simplex method

This is part of the first exercise (Opgave 1) in the exam of 2008
Consider the following linear programming problem (P1)

$$\begin{aligned} \text{maximize } & 2x_1 + 4x_2 - x_3 \\ \text{subject to } & 2x_1 - x_3 \leq 6 \\ & 3x_2 - x_3 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- Rewrite the problem in equational standard form adding the slack variables x_4, x_5, x_6 to the three constraints above, respectively, and write the first simplex tableau with x_4, x_5, x_6 as basis solution.
- Argue that x_2 can be brought in the basis with advantage and perform one pivot iteration that brings x_2 into the basis solution.
- After another pivot iteration, it is x_1 that can be brought with advantage in the basis (you do not have to perform this iteration), reaching the following simplex tableau:

```
|--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -5/3 |      1 |    2/3 |     -2 |      0 |      4 |
|      0 |      1 |   -1/3 |      0 |    1/3 |      0 |      0 |      3 |
|      1 |      0 |    1/3 |      0 |   -1/3 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -1/3 |      0 |   -2/3 |     -2 |      1 |    -14 |
|--------+--------+--------+--------+--------+--------+--------+--------+
```

  Argue that an optimal solution is found and give the solution together with its objective value.

Exercise 2* Simplex method

Solve the following LP problem carrying out the simplex operations by hand:

$$\begin{aligned} \text{maximize } & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to } & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

You are free to use any of the two representations, tableau or dictionary.
You can also get help from Python. You find a tutorial for what you need at this link:

http://www.imada.sdu.dk/~marco/DM545/Resources/Ipython/Tutorial4Exam.html.

Exercise 3

Solve the following linear programming problem applying the simplex algorithm:

$$
\begin{array}{ll}
\text{maximize} & 3x_1 + 2x_2 \\
\text{subject to} & x_1 - 2x_2 \leq 1 \\
& x_1 - x_2 \leq 2 \\
& 2x_1 - x_2 \leq 6 \\
& x_1 \leq 5 \\
& 2x_1 + x_2 \leq 16 \\
& x_1 + x_2 \leq 12 \\
& x_1 + 2x_2 \leq 21 \\
& x_2 \leq 10 \\
& x_1, x_2 \geq 0.
\end{array}
$$

[Hint: you can plot the feasibility region with one of the tools linked at the course web page: "Tools" -> "Web applications on the simplex" -> "LP Simplex" and use the clairvoyant's rule to minimize the number of operations to carry out.]

Exercise 4*

Consider the following problem:

$$
\begin{array}{rl}
\max & z = 4x_2 \\
\text{s.t.} & 2x_2 \geq 0 \\
& -3x_1 + 4x_2 \geq 1 \\
& x_1, x_2 \geq 0
\end{array}
$$

a. Write the LP in standard form (that is, in equation form with slack or surplus variables) and say why it does not provide immediately an initial feasible basis for the simplex method.

Exercise 5

Solve the following problem, known as the Klee-Minty problem, using the largest coefficient pivoting rule.

$$
\begin{array}{ll}
\text{maximize} & 100x_1 + 10x_2 + x_3 \\
\text{subject to} & x_1 \leq 1 \\
& 20x_1 + x_2 \leq 100 \\
& 200x_1 + 20x_2 + x_3 \leq 10000 \\
& x_1, x_2 \geq 0
\end{array}
$$

Can you generalize the example to n variables and guess what will be the number of iterations the simplex will do?

Exercise 6* Project Scheduling

[This exercise is a part of one that appeared in Exam 2011] A small project has 6 sub-activities A, B, C, D, E, F whose individual dependency (shown by the immediate predecessors) is given in Figure 1. Here we also list the normal time (in weeks), the absolute minimum time and the cost of shortening the activity by one week.
The goal is to shorten the duration of the project to 19 weeks. This means that the duration of one or more activities has to be shortened. Of course we want to select these so that the total cost of shortening the duration to 19 weeks is minimized. Formulate this problem as a linear programming problem and argue that the optimal solution to this LP will provide the correct answer. Note that you must use the actual data in the LP formulation!

Exercise 7 The pooling problem

[This exercise appeared in Exam 2013] A bartender serves usually alcoholic beverages behind the bar. A bartender can generally mix classic cocktails such as a Gin-and-Tonic, Caipirinha and Mojito. In order to achieve this task the bartender has to maintain the supplies and inventory for the bar.
End drinks for the customers are created by directly mixing the raw materials. Restricting our attention to the Gin-and-Tonic drink, the suggested ratios of gin and tonic are 1:1, 2:3, 1:2, and 1:3 (source Wikipedia). The historical data indicate nevertheless that the typical demand on a Saturday evening in the premise of our bartender is as follows:

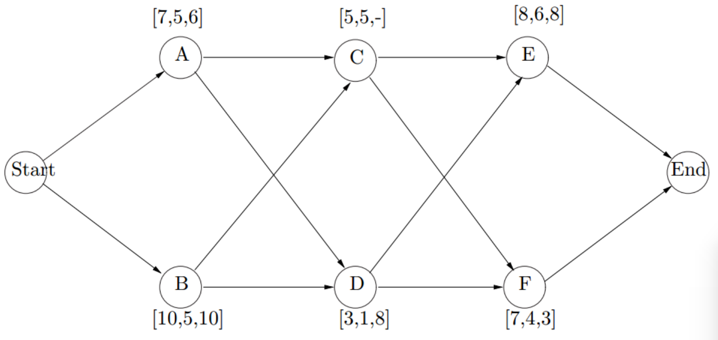

Figure 1: A network with activities on nodes for a small project with 6 activities. For each activity the following data is given in that order from left to right: normal time, minimum time in weeks, and the cost of shortening the duration of the activity by one week.

Alcohol	quantity	price per deciliter
$\geq 20\%$	≤ 12 l	30 dkk
$\geq 16\%$	≤ 12 l	25 dkk
$\geq 13\%$	≤ 12 l	21 dkk
$\geq 10\%$	≤ 12 l	15 dkk

For example, the first row indicates that there are customers that are willing to intake 20% or more alcohol consuming all together up to 12 liters and willing to pay 30 krone per deciliter.
Our bartender for economical and logistic issues can buy up to 10 liters of gin that contain 40% alcohol and up to 20 liters of tonic that contain no alcohol.
The mixing process occurs in a way such that at the end the input products no longer exist in their original forms, but are mixed to form new mixed products with new property values, in this case, the percentage of alcohol content.
The objective is to maximize the total profit from the sell.
The problem is an example of blending problem that arises, beside bar keeping also in the oil refinery industry.
Represent the situation as a network and model the problem as a linear programming problem.