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DM559/DM545 – Linear and integer programming

Sheet 4, Spring 2018 [pdf format]

Starred exercises are relevant for the exam.

Exercise 1*

Given the polyhedron in standard form characterized by the following matrices A and b:

$$A = \begin{bmatrix} 2 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

List all vertices of the polyhedron.

Exercise 2*

The two following LP problems lead to two particular cases when solved by the simplex algorithm. Identify these cases and characterize them, that is, give indication of which conditions generate them in general.

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_2 \leq 5 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

Exercise 3

Solve the following LP problem

$$\begin{array}{ll} \text{maximize} & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & x_1 \leq 1 \\ & 1/2x_1 - 11/2x_2 - 5/2x_3 + 9x_4 \leq 0 \\ & +1/2x_1 - 3/2x_2 - 1/2x_3 + x_4 \leq 0 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

using the following pivot rule:

i. the entering variable will always be the nonbasic variable that has the largest coefficient in the z-row of the dictionary.

ii. if two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave.

There are quite a few calculations to carry out, hence you are reccomended to use Python. See the guidelines at: http://www.imada.sdu.dk/~marco/DM545/Resources/Ipython/Tutorial4Exam.html

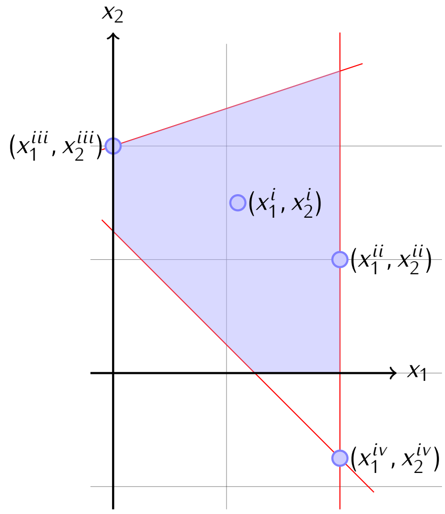
Exercise 4

What argument is used to prove that the simplex algorithm always terminates in a finite number of iterations if it does not encounter a situation in which one of the basic variables is zero? What may happen instead if the latter situation arises and which remedies are introduced?

Exercise 5* Exercise 3 from Exam 2013

Consider the following LP problem:

$$
\begin{array}{llrcrl}
(P) & \max & z = x_1 & + & 5x_2 & \\
& \text{s.t.} & -x_1 & + & 3x_2 & \leq 6 \\
& & 4x_1 & + & 4x_2 & \geq 5 \\
& & 0 & \leq & x_1 & \leq 2 \\
& & & & x_2 & \geq 0
\end{array}
$$



(Note: the following subtasks can be carried out independently; use fractional mode for numerical calculations; in the online version you find the problem in ASCII format).

a. The polyhedron representing the feasibility region is depicted in the figure. Indicate for each of the four points represented whether they are feasible and/or basic solutions. Justify your answer.

b. Write the initial tableau or dictionary for the simplex method. Write the corresponding basic solution and its value. State whether the solution is feasible or not and whether it is optimal or not.

c. Consider the following tableau:

```
|-----+----+----+----+------+----+----+------|
|     | x1 | x2 | x3 | x4   | x5 | -z | b    |
|-----+----+----+----+------+----+----+------|
| I   |  0 |  4 |  1 | -1/4 |  0 |  0 | 29/4 |
| II  |  1 |  1 |  0 | -1/4 |  0 |  0 | 5/4  |
| III |  0 | -1 |  0 | 1/4  |  1 |  0 | 3/4  |
| IV  |  0 |  4 |  0 | 1/4  |  0 |  1 | -5/4 |
|-----+----+----+----+------+----+----+------|
```

and the following three pivoting rules:

- largest coefficient
- largest increase
- steepest edge.

Which entering and leaving variables would each of them indicate? In this specific case, which rule would be convenient to follow? Report the details of the computations for the first two rules and carry out graphically the application of the third rule using the plot in the figure above (tikz code to reproduce the figure available in the online version.)

Exercise 6*

Consider the following problem:

$$
\begin{aligned}
\max\ & z = 4x_2 \\
\text{s.t.}\ & 2x_2 \geq 0 \\
& -3x_1 + 4x_2 \geq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

a. Write the LP in equational standard form and say why it does not provide immediately an initial feasible basis for the simplex method.

b. To overcome the situation of infeasible basis construct the auxiliary problem for a phase I-phase II solution approach. Determine which variables are initially in basis and which are not in basis in the auxiliary problem.

c. Answer the following questions

i) Is the initial basis in the auxiliary problem feasible in the original problem?
ii) Is it optimal in the auxiliary problem?
iii) Is it degenerate?
iv) Can we say at this stage if phase I will terminate?
v) If it will terminate, can we say at this stage that it will terminate with a basis that corresponds to a feasible solution in the original problem?
vi) Solve the problem by carrying out Phase I and Phase II of the simplex algorithm.

Exercise 7*

Show that the dual of $\max\{c^T x \mid Ax = b, x \geq 0\}$ is $\min\{y^T b \mid y^T A \geq c\}$.

Exercise 8*

Consider the following LP problem:

$$\begin{array}{ll} \max & 2x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 30 \\ & x_1 + 2x_2 \geq 10 \\ & x_1 - x_2 \leq 1 \\ & x_2 - x_1 \leq 1 \\ & x_1 \geq 0 \end{array}$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Exercise 9*

Consider the problem

$$\begin{array}{ll} \text{maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]