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Sheet 6, Spring 2017 [pdf format]

Exercise 1*

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- In lecture 4 we saw a way to find an initial feasible basis via an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. Phase I is thus carried out by solving an auxiliary LP problem whose solution gives an initial feasible basis or a proof of infeasibility.
- The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:
  1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
  2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| : a_{i^*j} < 0\}$
  3. Update the tableau around the pivot in the same way as with the primal simplex.
  4. Stop if no right-hand side term is negative.

Opposite to the primal simplex method, the dual simplex method iterates through infeasible basis solutions, while maintaining them optimal, and stops when a feasible solution is reached.
Duality can help us with the issue of initial feasible basis solutions. In the problem above, if the objective function was $w = -x_1 - x_2$, then the initial basis solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. But with objective function z the simplex has infeasible initial basis in both problems. However we can change temporarily the objective function z with w and apply the dual simplex method. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and continue iterating with the primal simplex. This phase I–phase II simplex method is also called the *dual-primal simplex method*. Apply this method to the problem above and verify that it leads to the same solution as in point 1.

Exercise 2* Sensitivity Analysis and Revised Simplex

A furniture-manufacturing company can produce four types of product using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.

- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.
- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve the following LP problem

$$\begin{aligned} \max \quad & 19x_1 + 13x_2 + 12x_3 + 17x_4 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 \\ & x_1 + x_2 + x_3 + x_4 \leq 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The final tableau has x_1, x_3 and x_4 in basis. With the help of a computational environment such as Python for carrying out linear algebra operations, address the following points:

a) Write A_B, A_N, $A_B^{-1}A_N$, the final simplex tableau and verify that the solution is indeed optimal.

b) What is the increase in price (reduced cost) that would make product x_2 worth to be produced?

c) What is the marginal value (shadow price) of an extra hour of work or amount of metal and wood?

d) Are all resources totally utilized, i.e. are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

e) From the economical interpretation of the dual why product x_2 is not worth producing? What is its imputed cost?

Solve the following variations:

1. The net profit brought in by each desk increases from 13 Euro to 15 Euro.
2. The availability of metal increases from 117 to 125 units per day
3. The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood and bring in a net profit of 14 Euro.
4. The number of chairs produced must be at most five times the numbers of desks

Exercise 3

Solve the systems $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1\ 2\ 3]$ and $E_1 E_2 E_3 E_4 \mathbf{d} = [1\ 2\ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Exercise 4* Quzzies

Basic Geometric Facts

1. In 4D, how many hyperplanes need to intersect to give a point?
2. In 4D, can a point be described by more than 4 hyperplanes?
3. Consider the intersection of n hyperplanes in n dimensions: when does it uniquely identify a point?

Vertices of Polyhedra:
Consider the polyhedron described by $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$, that is:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\ & & \vdots & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m \end{array}$$

4. How many constraints are *active* in a *vertex* of a polyhedron $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$?

5. Does every point x that activates n constraints form a vertex of the polyhedron?

6. Can a vertex activate more than n constraints?

7. What if there are more variables than constraints? If $m > n$ then we can find a subset and then activate but what if $m < n$, can we have a vertex?

8. Combinatorial explosion of vertices: how many constraints and vertices has an n-dimensional hypercube?

9. If there are m constraints and n variables, $m > n$, what is an upper bound to the number of vertices?

Tableaux and Vertices

10. For each of these three statements, say if they are true or false:

   – One tableau $\Longrightarrow$ one vertex of the feasible region

   – One tableau $\Longleftarrow$ one vertex of the feasible region

   – One tableau $\Longleftrightarrow$ one vertex of the feasible region

11. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
x_1	1	0	−1/5	1/2	0	8
	0	0	−2/5	−1	1	−64

   – How many variables (original and slack) can be different from zero?

   – $(x_3, x_4) = (0, 0)$ are non basic, what does this tell us about the constraints?

   Let's generalize the previous case. Consider an LP with m constraints, n original variables and m slack variables. In an optimal solution:

   – is $m > n$, how many variables (original and slack) can be nonzero at most?

   – If $m < n$ how many original variables must be zero at least? In other terms, in a mix planning problem with n products and m, $m < n$ resources, how many products at most will be to be produced in an optimal solution?

12. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	0	1	1/2	0	1
x_1	1	1	0	−1/2	0	1
	0	−2	0	1/2	1	−1

   $(x_2, x_4) = (0, 0)$ is non basic, what does this tell us about the constraints?

13. If in the original space of the problem we had 3 variables, and there are 6 constraints, how many constraints would be active?

14. For the general case with n original variables:
    One basic feasible solution $\iff$ a matrix of active constraints has rank n. True or False?

15. Consider an LP problem with m constraints and n original variables, $m > n$. We saw that in $\mathbb{R}^n$ a point is the intersection of at least n hyperplanes. In LP this corresponds to say that in a vertex there are n active constraints. Let a tableau be associated with a solution that makes exactly $n + 1$ constraints active, what can we say about the corresponding basic and non-basic variable values?

16. What is the algebraic definition of adjacency in 2, 3 and n dimensions?

17. How does this condition translate in terms of tableau?