Department of Mathematics and Computer Science
University of Southern Denmark, Odense

March 31, 2017
Marco Chiarandini

DM559/DM545 – Linear and integer programming

Sheet 1, Spring 2017 [pdf format]

Solution:

Included.

For an example of what is expected as an answer for these exercises about modeling you should take a look at Sec. 1.3.2 from the Lecture Notes [LN] on the resource allocation problem. In particular, you should introduce the mathematical notation, give the linear programming model and finally explain each line of the model.

Note that at the exam your answers must be digitalized, hence it is good to start becoming acquainted with different tools to produce text documents containing mathematical notation and graphs. Read the "Instructions for Written Exam" in the Assessment section of the course web page for a list of useful tools.

Solution:

Recall that for modelling you have to determine in mathematical terms the following elements:

- Parameters
- Variables
- Objective function
- Constraints.

The description of the model has to be organized in the following parts:

- Introduction of the mathematical notation: which symbols denote parameters and variables? which indices are you using and where are they running?
- Mathematical model:

$$\begin{aligned} \max \quad & \mathbf{c}^t\mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

  (Most likely you will express the model in scalar notation. Remember not to leave any index unspecified. Watch out the quantifiers.)
- Explanation of each line of the mathematical model.

Remember to check that every symbol and index in your model is defined.

LP Modeling

Exercise 1* Food manufacture

A chocolate factory produces two types of chocolate bars: Milk & Hazelnuts (M&H) chocholate and Dark (D) chocholate. The price for a pack of M&H and a pack of D is 100 and 160 Dkk, respectively. Each pack of M&H is made of 50 grams of hazelnuts, 60 grams of chocolate, 40 grams of milk and 50 grams of sugar. Each Dark bar contains 150 grams of chocolate, 50 grams of fat and 30 grams of sugar. The factory has 200 grams of hazelnuts, 1000 grams of chocolate, 250 grams of milk, 300 grams of sugar, and 300 grams of fat left.

Your goal is to determine how many M&H bars and D bars the company should produce to maximize its profit.

a) Give the linear program for the problem, using variables x_1 and x_2 and the parameters defined above. Specify (i) the constraints, and (ii) the objective function.

b) Graph the feasible region of your linear program.

c) Compute the profit at each vertex of the feasible region, and report the best solution.

Solution:

The problem is a factory planning problem. We use the same notation introduced in class. hence I skip its definition here. The notation must otherwise always be introduced before the model.
Here we denote with the index 1 the M&H bars and with the index 2 the D bars. The variables x_1 and x_2 indicate the number of bars to be produced. We assume that this number is a divisible quantity, such that we can accept a result in terms of fractions of a pack.
The general model is:

$$
\begin{aligned}
\max \sum_{j=1}^{n} c_j x_j \\
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

If we introduce the given data we have.

$$
\begin{array}{rrcrcr}
\max & 100x_1 & + & 160x_2 & & \\
\text{nuts:} & 50x_1 & + & 0x_2 & \leq & 200 \\
\text{cacao:} & 60x_1 & + & 150x_2 & \leq & 1000 \\
\text{milk:} & 40x_1 & + & 0x_2 & \leq & 250 \\
\text{sugar:} & 50x_1 & + & 30x_2 & \leq & 300 \\
\text{fat:} & 0x_1 & + & 50x_2 & \leq & 300 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Each constraint represents an halfplane (an halfspace in 2D) defined by the line, whose form is described by the equation of the inequality. For example the first halfplane is the space below the line described by

$$60x_1 + 150x_2 = 1000$$

We can graph all these lines and indicate the feasible region in the Cartesian plane since the problem has only two variables. We can do this by hand or using any avialble tool for graphing mathematical expressions. From the course web page you find the link to LP Grapher, an online app for graphing LP problems. Inputting our data in that app we obtain Figure 1.
It would be also possible to plot the objective function. This is the line expressed by the parametric equation:

$$100x_1 + 160x_2 = k$$

where k is a parameter that defines the value of a solution. Since we know that a solution is in a vertex of the polytope, it makes sense to calculate the objective function at each vertex. The vertex where the optimal solution is is indicated in the figure.

Exercise 2* Optimal Blending

The Metalco Company wants to blend a new alloy (metal) made by 40 percent tin, 35 percent zinc, and 25 percent lead from 5 available alloys having the following properties:
The objective is to determine the proportions of these alloys that should be blended to produce the new alloy at a minimum cost. Formulate a linear programming model for this problem. [Problem from ref. HL]
To help you in the task we start introducing the mathematical notation that will be used for the model. Let $J = \{1, 2, \ldots, 5\}$ indexed by j be the set of alloys and $I = \{\text{tin}, \text{zinc}, \text{lead}\}$ indexed by i be the set of metals. Let a_{ij} be the fixed parameters that determine the percentage amount of metal i in alloy j. Let c_j be the cost of alloy j in Dkk per Kg. The problem asks to determine the proportions of the alloys to

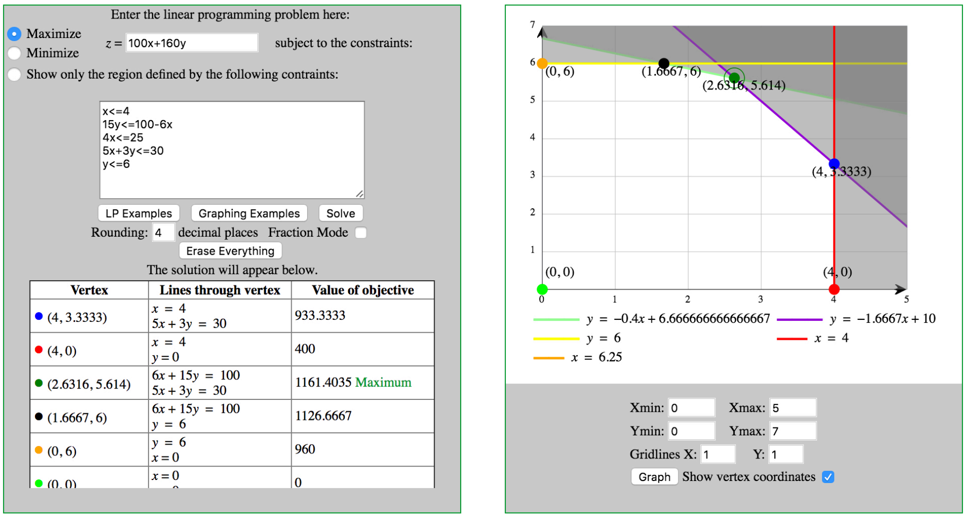

Figure 1: LP grapher for Exercise .

	Alloy				
Property	1	2	3	4	5
Percentage of tin	60	25	45	20	50
Percentage of zinc	10	15	45	50	40
Percentage of lead	30	60	10	30	10
Cost (DKK/Kg)	77	70	88	84	94

blend to obtain the new alloy with the properties of 40% tin, 35% zinc and 25% lead. Let's call these last parameters $b_i, i \in I$ and the proportion of each alloy to blend with respect to the new alloy by $y_j \geq 0$. Specify the constraints and the objective function using the mathematical terms introduced.
[Units of measure offer a way to test the correctness of the model. In particular, the proportion y_j can be seen as the amount of alloy in Kg per amount of new alloy. Let α be the amount in Kg of the new alloy, and let x_j be the amount in Kg of each alloy from I. Then $y_j = x_j/\alpha$ and it is an adimensional quantity.]

Solution:

If we denote by α the amount in Kg of the new alloy, the blend will satisfy the following equation as far as tin is concerned:

$$\frac{60}{100}x_1 + \frac{25}{100}x_2 + \frac{45}{100}x_3 + \frac{20}{100}x_4 + \frac{50}{100}x_5 = \frac{40}{100}\alpha$$

For the other two components, zinc and lead, the equations will be similar. The value of α is positive and arbitrary. We can divide LHS and RHS by α. Substituting $y_j = x_j/\alpha$ we obtain the amount of alloy per Kg of the new alloy, that is, the proportion we want to determine. We can then find the values of the variables y_j by solving the following LP problem:

$$\min \sum_{j \in J} c_j y_j \qquad (1)$$

$$\sum_{j \in J} a_{ij} y_j \geq b_i \qquad \forall i \in I \qquad (2)$$

$$y_j \geq 0 \qquad \forall j \in J \qquad (3)$$

The objective function (1) calculates the total cost of the alloy. Note that the total cost would be $c_j x_j$, where c_j is expressed in Dkk/Kg and x_j in Kg. If we use y_j in place of x_j we should interpret the total cost as referring to $\alpha = 1$ Kg of the new alloy. The set of three constraints (2)-(3) express the result of the blend.

Exercise 3*

A cargo plane has three compartments for storing cargo: front, center and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	Weight Capacity (Tons)	Space Capacity (Cubic meters)
Front	12	7000
Center	18	9000
Back	10	5000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.
The following four cargos have been offered for shipment on an upcoming flight as space is available:

Cargo	Weight(Tons)	Volume (Cubic meters/Tons)	Profit
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Formulate a linear programming model for this problem. [Problem from ref. HL]

Solution:

Let $I = \{1, 2, 3\}$ be the set of compartments in the plane and $J = \{1, 2, 3, 4\}$ the set of cargos. Let's denote by w_j and v_j the weight and the volume of cargo $j \in J$, respectively and by W_i and V_j the capacities of the compartments. Let also p_j be the profit per ton of cargo j. This quantity is obtained by dividing the profit of the whole cargo by its weight. For example, for cargo 1, $p_1 = 320/20$. Finally, let η_i be the proportion of a compartment's weight capacity with repect to the total capacity of the airplane:

$$\xi_i = \frac{V_i}{\sum_{i \in I} V_i} \qquad \forall i \in I$$

The task is to decide the proportion of each cargo to load in each compartment. Hence, we introduce a variable $x_{i,j}$ for each compartment $i \in I$ and each cargo $j \in J$ representing the proportion of the cargo j that goes in compartment i. Since $x_{i,j}$ is a proportion its value has to be in the closed interval $[0, 1]$. We will also need an auxiliary variable y_i to represent the proportion of weight put in compartment $i \in I$.
The model is:

$$\max \sum_{j \in J} p_j \sum_{i \in I} w_j x_j \tag{4}$$

$$\sum_{j \in J} w_j x_{ij} \leq W_i \qquad \forall i \in I \tag{5}$$

$$\sum_{j \in J} v_j x_{ij} \leq V_i \qquad \forall i \in I \tag{6}$$

$$\sum_{j \in J} x_{ij} = y_i \qquad \forall i \in I \tag{7}$$

$$\xi_i \sum_{i \in i} y_i = y_i \qquad \forall i \in I \tag{8}$$

$$y_i \geq 0 \qquad \forall i \in I \tag{9}$$

$$x_{ij} \geq 0 \qquad \forall j \in J, \forall i \in I \tag{10}$$

$$x_{ij} \leq 1 \qquad \forall j \in J, \forall i \in I \tag{11}$$

The objective function (4) maximizes the total profit. Constraints (5) and (6) ensure that the capacity limits for each compartment are not exceeded. Constraints (7) make the link between the variables x_{ij} and the variables y_i. Constraints (8) impose that the proportions of cargos in the compartment match the proportion of capacities. Finally, constrations (9), (10), (11) define the domain of the variables.

Exercise 4*

A small airline flies between three cities: Copenhagen, Aarhus, and Odense. They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Copenhagen, stops in Odense, and continues to Aarhus. There are three types of passengers:

- (a) Those traveling from Copenhagen to Odense.
- (b) Those traveling from Odense to Aarhus.
- (c) Those traveling from Copenhagen to Aarhus.

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

- (a) Y class: full coach.
- (b) B class: nonrefundable.
- (c) M class: nonrefundable, 3-week advanced purchase.

Ticket prices, which are largely determined by external influences (i.e., train and bus competitors), have been set and advertised as follows:

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	300	160	360
B	220	130	280
M	100	80	140

Based on past experience, demand forecasters at the airline have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	4	8	3
B	8	13	10
M	22	20	18

The goal is to decide how many tickets from each of the 9 origin/destination/ fare-class combinations to sell. The constraints are that the plane cannot be overbooked on either of the two legs of the flight and that the number of tickets made available cannot exceed the forecasted maximum demand. The objective is to maximize the revenue.
Formulate this problem as a linear programming problem.

Solution:
Let $I = \{CO, OD, CA\}$ indexed by i be the set of legs and $J = \{Y, B, M\}$ indexed by j the set of fares. Let p_{ij} the prices of the tickets in the various categories and flights.
The decision variables are x_{ij} for $i \in I$ and $j \in J$ and indicate the number of tickets to tickets to sell in each category.

$$\max \sum_{i \in I} \sum_{j \in J} p_{ij} x_{ij} \tag{12}$$

$$\sum_{j \in J} (x_{CO,j} + X_{CA,j}) \leq 30 \tag{13}$$

$$\sum_{j \in J} (x_{OA,j} + X_{CA,j}) \leq 30 \tag{14}$$

$$x_{ij} \leq u_{ij} \qquad \forall i \in I, \forall j \in J \tag{15}$$

$$x_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J \tag{16}$$

The first constraints ensures that the flight is not overbooked in the leg C to O, while the second constraint ensures that the flight is not overbooked in the leg O-A.

Exercise 5

Suppose that Y is a random variable taking on one of n known values:

$$a_1, a_2, \ldots, a_n$$

Suppose we know that Y either has distribution p given by

$$P(Y = a_j) = p_j$$

or it has distribution q given by

$$P(Y = a_j) = q_j$$

Of course, the numbers $p_j, j = 1, 2, \ldots, n$ are nonnegative and sum to one. The same is true for the q_j's. Based on a single observation of Y, we wish to guess whether it has distribution p or distribution q. That is, for each possible outcome a_j, we will assert with probability x_j that the distribution is p and with probability $1 - x_j$ that the distribution is q. We wish to determine the probabilities $x_j, j = 1, 2, \ldots, n$, such that the joint probability of saying the distribution is p when in fact it is q over the outcomes $j = 1, \ldots, n$ is no larger than β, where β is some small positive value (such as 0.05). Furthermore, given this constraint, we wish to maximize the probability that we say the distribution is p when in fact it is p. Formulate this maximization problem as a linear programming problem.

Solution:

With probability x_j we say the distribution is p
with probability $1 - x_j$ we say the distribution is q
The joint probability of saying that the distribution is p when it is p is given by the product rule: ie, for two independent events A and B the joint probability $p(A|B)$ is given by $p(AB) = p(A)p(B)$. For each single toss, the expected probability of saying wrong is the sum over j of the joint probabilities. We find x_j by solving:

$$\max \sum_{j=1}^{n} p_j x_j \tag{17}$$

$$\sum_{j=1}^{n} q_j x_j \leq \beta \tag{18}$$

$$x_j \geq 0 \qquad \forall j = 1, \ldots, n \tag{19}$$

$$x_j \leq 1 \qquad \forall j = 1, \ldots, n \tag{20}$$

The objective (17) maximizes the probability to say correct. It uses again the law of joint probabilities. Constraint (18) ensures that we say wrong with probability no more than β and constraints (20) and (??) ensure that the valeus x_j satisfy the axioms of probabilities.

Exercise 6

In this exercise we study the application of linear programming to an area of statistics, namely, regression.

Consider a set of $m = 9$ measurements: $28, 62, 80, 84, 86, 86, 92, 95, 98$. A way to summarize these data is by their mean

$$\bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i$$

An alternative way is by the median, ie, the measurement that is worse than half of the other scores and better than the other half.
There is a close connection between these statistics and optimization. Show that the mean is the measure that minimizes the sum of squared deviation between the data points and itself and that the median minimizes the sum of the absolute values of the differences between each data point and itself.

Consider now a set of points on a two-dimensional space $S = (x_1, y_1), \ldots, (x_m, y_m)$. The points are measurements of a response variable given some control variable, for example, blood pressure given the weight of a person. The points hint at a linear dependency between the variables representing the two dimensions. We may assume a random fluctuation around the right value and hence the following regression model:

$$y = ax + b + \epsilon$$

Specifically, for our set of points S we have

$$y_i = ax_i + b + \epsilon_i, \quad i = 1, \ldots, m$$

We thus want to find a line that best fits the measured points. That is, we wish to determine the (unknown) numbers a and b. There is no unique criterion to formulate the desire that a given line "best fits" the points. The task can be achieved by minimizing, in some sense, the vector ϵ. As for the mean and median, we can consider minimizing either the sum of the squares of the ϵ_i's or the sum of the absolute values of the ϵ_i's. These concepts are formalized in measure theory by the so called L^p-norm ($1 \leq p < \infty$). For the vector ϵ:

$$\epsilon_{\ p} = \left(\sum_i \epsilon_i^p \right)^{1/p}$$

The method of least squares, which is perhaps the most popular, corresponds to L^2-norm. The minimization of L^2-norm for the vector ϵ in the variables a and b has a closed form solution that you may have encountered in the statistics courses. This method needs not always to be the most suitable, however. For instance, if a few exceptional points are measured with very large error, they can influence the resulting line a great deal. Just as the median gives a more robust estimate of a collection of numbers than the means, the L^1 norm is less sensitive to outliers than least square regression is. The problem is to solve the following minimization problem:

$$\text{argmin}_{a,b} \sum_i |\epsilon_i| = \text{argmin}_{a,b} \sum_i |ax_i + b - y_i|$$

Unlike for least square regression, there is no explicit formula for the solution of the L^1-regression problem. However the problem can be formulated as a linear programming problem. Show how this can be done.

The regression via L^∞-norm corresponds to solve the problem:

$$\text{argmin} \max_{i=1}^{n} |ax_i + b - y_i|$$

This problem can also be solved by linear programming. Show how to formulate the problem as a linear program.

Solution:

We consider first the L_1-norm.
The problem we want to solve is:

$$\min_{a,b} \sum_{i=1}^{n} |ax_i + b - y_i|$$

There are two ways we can rewrite this problem as an LP problem.
In the first, we introduce n new variables, one for each term in the summation, ie, $z_i \in \mathbb{R}$, $i = 1..n$ and write:

$$\begin{array}{lrcll} \min & \sum_{i=1}^{n} z_i & & & \\ \text{s.t.} & z_i & \geq & ax_i + b - y_i & i = 1..n \\ & z_i & \geq & -(ax_i + b - y_i) & i = 1..n \\ & z_i & \in & \mathbb{R} & i = 1..n \\ & a, b & \in & \mathbb{R} & \end{array}$$

If the ith term is positive then the value of the corresponding z_i variable is determined by the first constraint, if it is negative, then the value is determined by the second constraint. Hence, we obtained an LP model with n additional variables and $2n$ additional constraints.

The second approach consists in introducing two non-negative variables z_i^+ and z_i^- for each ith term. The LP model is then:

$$\begin{array}{rrcll} \min & \sum_{i=1}^{n}(z_i^+ + z_i^-) & & & \\ \text{s.t.} & ax_i + b - y_i & = & z_i^+ - z_i^- & i = 1..n \\ & z_i^+, z_i^- & \geq & 0 & i = 1..n \\ & a, b & \in & \mathbb{R} & \end{array}$$

Since both terms on the right hand side of the first constraint are nonnegative, if the left hand side is positive then z_i^+ will take all the value (any other arrangement would give an higher correspoding value in the objective function for the ith term). If the left hand side is negative then the value will go all in z_i^-. Hence, we obtained an LP problem with $2n$ additional variables and n additional constraints. Empirical experiments can determine which of the two approaches is the best.

For the L_∞-norm, we want to solve the following problem:

$$\min_{a,b}\left[\max_{i=1..n}\{|ax_i + b - y_i|\}\right]$$

This problem is not in linear form but it cane be reduced to a linear form as follows. We introduce one new variable z and write:

$$\begin{array}{rrcll} \min & z & & & \\ \text{s.t.} & z & \geq & ax_i + b - y_i & i = 1..n \\ & z & \geq & -(ax_i + b - y_i) & i = 1..n \\ & z & \in & \mathbb{R} & \\ & a, b & \in & \mathbb{R} & \end{array}$$

Note that this corresponds to setting

$$z \geq \max\{ax_i + b - y_i, -(ax_i + b - y_i)\}$$

To obtain the LP model we have introduced 1 additional variable and $2n$ constraints.