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DM559/DM545 – Linear and integer programming

Sheet 4, Spring 2018 [pdf format]

Starred exercises are relevant for the exam.

Solution:
Included. The HTML may not be well formatted. See PDF version.

Exercise 1* Given the polyhedron in standard form characterized by the following matrices A and b:

$$A = \begin{bmatrix} 2 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

List all vertices of the polyhedron.

Solution:
The polyhedron in standard form is defined by $Ax \leq b$. Let $\bar{A}$ be the matrix after the introduction of the slack variables, ie,

$$\bar{A} = \begin{bmatrix} 2 & 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let m be the number of constraints and n the number of variables of the system of linear equations $\bar{A}\bar{x} = b$. Hence, $\bar{A} \in \mathbb{R}^{m \times n}$, $\bar{x} \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Recall that the system admits solutions if and only if: $\text{rank}(\bar{A}) = \text{rank}(\bar{A} \mid b)$.
If $n < m$ and hence $\text{rank}(\bar{A}) \leq n$ the system is overdetermined and likely infeasible, unless $\text{rank}(\bar{A}) = \text{rank}(\bar{A} \mid b)$, which would imply that the constraints are linearly dependent. Linearly dependent constraints can be removed as they are redundant. Thus, we can assume that $n \geq m$ and most likely $n > m$ since we introduced a slack variable for each constraint. Moreover, since we removed linearly dependent constraints: $\text{rank}(\bar{A}) = m$ and consequently it must be $\text{rank}(\bar{A}) = \text{rank}(\bar{A} \mid b)$. Under the assumption that $n > m$ then the system is underdetermined. The solutions have $n - m$ free variables and the solution space has dimension $n - m$. It is the simplex represented by the intersection of $\bar{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. The vertices of this simplex in $\mathbb{R}^n$ are the vertices of the polyhedron described by A and b. Algebraically, they correspond to the basic feasible solutions of the linear system $\bar{A}\bar{x} = b$. This means that to find the vertices of the polyhedron we need to enumerate all basic solutions of $\bar{A}\bar{x} = b$. We recall that a basic solution is given by a subset B of size m of the indices of columns of the matrix $\bar{A}$ and is such that A_B, the so-called basis matrix, is non-singular and $\bar{x}_B = A_B^{-1}b \geq 0$ and $\bar{x}_N = 0$. To determine all bases we need to generate all combinations of size $m = 3$ of the $n = 7$ columns. We write a python script to do the calculations for us:

```
import scipy as sc
import scipy.linalg as sl
import sympy as sy
import itertools as it

A = sc.array([[2,0,1,-4],[0,1,0,2],[0,0,1,0]])
I = sc.identity(3)
A = sc.concatenate([A,I],axis=1)
print A
b = sc.array([3,1,1])

for e in it.combinations(range(7),3):
    if sl.det(A[:,e]) != 0:
```

```
        x = sc.dot(sl.inv(A[:,e]),b)
        if (x>=0).all:
            print e,x
        else:
            print e," infesible"
    else:
        print e, sy.Matrix(sc.column_stack([A[:,e],b])).rref()
```

Exercise 2*

The two following LP problems lead to two particular cases when solved by the simplex algorithm. Identify these cases and characterize them, that is, give indication of which conditions generate them in general.

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_2 \leq 5 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution:

In the initial tableau of the first problem there is a column, the first one, with positive reduced cost and no positive a_{ij} term. This means that the corresponding variable x_1 can be brought into the basis but the increase of its value is unlimited. This indicates that we have an unbounded problem.
The second LP problem is developed in the slides for the lecture on exception handling. After some iterations we reach a tableau in which a non basic variable has reduced cost zero. This indicates that it can be brought in the basis without a change in the objective function. Since the solution changes when we bring the variable in basis then the problem has more than one solution and it has therefore infinite solutions. They can be expressed as the convex combination of all optimal basic soltuions.

Exercise 3

Solve the following LP problem

$$\begin{array}{ll} \text{maximize} & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & x_1 \leq 1 \\ & 1/2x_1 - 11/2x_2 - 5/2x_3 + 9x_4 \leq 0 \\ & +1/2x_1 - 3/2x_2 - 1/2x_3 + x_4 \leq 0 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

using the following pivot rule:

i. the entering variable will always be the nonbasic variable that has the largest coefficient in the z-row of the dictionary.

ii. if two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave.

There are quite a few calculations to carry out, hence you are reccomended to use Python. See the guidelines at: http://www.imada.sdu.dk/~marco/DM545/Resources/Ipython/Tutorial4Exam.html

Solution:

```
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     x7 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|    1/2 |  -11/2 |   -5/2 |      9 |      1 |      0 |      0 |      0 |      0 |
|    1/2 |   -3/2 |   -1/2 |      1 |      0 |      1 |      0 |      0 |      0 |
|      1 |      0 |      0 |      0 |      0 |      0 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     10 |    -57 |     -9 |    -24 |      0 |      0 |      0 |      1 |      0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
PRIMAL SIMPLEX

pivot column: 1
pivot row: 1
 pivot: 1/2


|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     x7 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      1 |    -11 |     -5 |     18 |      2 |      0 |      0 |      0 |      0 |
|      0 |      4 |      2 |     -8 |     -1 |      1 |      0 |      0 |      0 |
|      0 |     11 |      5 |    -18 |     -2 |      0 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |     53 |     41 |   -204 |    -20 |      0 |      0 |      1 |      0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 2
pivot row: 2
 pivot: 4

|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     x7 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      1 |      0 |    1/2 |     -4 |   -3/4 |   11/4 |      0 |      0 |      0 |
|      0 |      1 |    1/2 |     -2 |   -1/4 |    1/4 |      0 |      0 |      0 |
|      0 |      0 |   -1/2 |      4 |    3/4 |  -11/4 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   29/2 |    -98 |  -27/4 |  -53/4 |      0 |      1 |      0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 3
pivot row: 1
 pivot: 1/2

|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|     x1 |     x2 |     x3 |     x4 |     x5 |     x6 |     x7 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|      2 |      0 |      1 |     -8 |   -3/2 |   11/2 |      0 |      0 |      0 |
|     -1 |      1 |      0 |      2 |    1/2 |   -5/2 |      0 |      0 |      0 |
|      1 |      0 |      0 |      0 |      0 |      0 |      1 |      0 |      1 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+
|    -29 |      0 |      0 |     18 |     15 |    -93 |      0 |      1 |      0 |
|--------+--------+--------+--------+--------+--------+--------+--------+--------+

pivot column: 4
pivot row: 2
 pivot: 2
```

x1	x2	x3	x4	x5	x6	x7	-z	b
-2	4	1	0	1/2	-9/2	0	0	0
-1/2	1/2	0	1	1/4	-5/4	0	0	0
1	0	0	0	0	0	1	0	1
-20	-9	0	0	21/2	-141/2	0	1	0

```
pivot column: 5
pivot row: 1
 pivot: 1/2
```

x1	x2	x3	x4	x5	x6	x7	-z	b
-4	8	2	0	1	-9	0	0	0
1/2	-3/2	-1/2	1	0	1	0	0	0
1	0	0	0	0	0	1	0	1
22	-93	-21	0	0	24	0	1	0

```
pivot column: 6
pivot row: 2
 pivot: 1
```

x1	x2	x3	x4	x5	x6	x7	-z	b
1/2	-11/2	-5/2	9	1	0	0	0	0
1/2	-3/2	-1/2	1	0	1	0	0	0
1	0	0	0	0	0	1	0	1
10	-57	-9	-24	0	0	0	1	0

```
pivot column: 1
pivot row: 1
 pivot: 1/2
```

Thus we discover that we return to the first tableau and that therefore we are cycling. We are in a malignous degenerancy. In order to make it benignous, that is, in order to avoid cycling a different pivoting rule must be used. The sign that we are in a degenerate case that might turn out malignous is the fact that one of the b_i terms is zero. This implies that there is a basic variable that gets value zero.

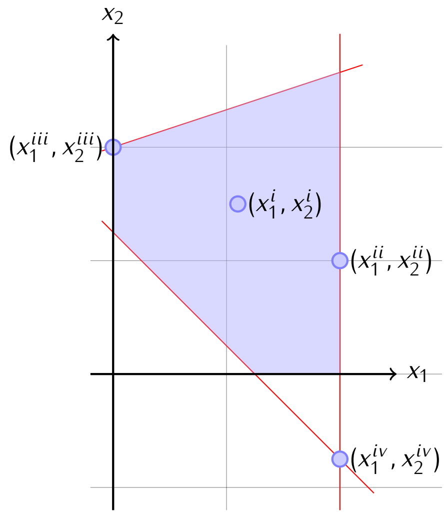
Exercise 4

What argument is used to prove that the simplex algorithm always terminates in a finite number of iterations if it does not encounter a situation in which one of the basic variables is zero? What may happen instead if the latter situation arises and which remedies are introduced?

Exercise 5* Exercise 3 from Exam 2013

Consider the following LP problem:

$$(P) \quad \begin{array}{rrcrl} \max & z = x_1 & + & 5x_2 & \\ \text{s.t.} & -x_1 & + & 3x_2 & \leq 6 \\ & 4x_1 & + & 4x_2 & \geq 5 \\ & 0 \leq & x_1 & & \leq 2 \\ & & & x_2 & \geq 0 \end{array}$$



(Note: the following subtasks can be carried out independently; use fractional mode for numerical calculations; in the online version you find the problem in ASCII format).

a. The polyhedron representing the feasibility region is depicted in the figure. Indicate for each of the four points represented whether they are feasible and/or basic solutions. Justify your answer.

Solution:

- Point 1 is a feasible solution but not basic (no constraint is active in that point).
- Point 2 is a feasible solution but not basic (only one constraint is active while two are needed)
- Point 3 is a basic feasible solution
- Point 4 is a basic solution (combination of two active constraints) but non feasible.

b. Write the initial tableau or dictionary for the simplex method. Write the corresponding basic solution and its value. State whether the solution is feasible or not and whether it is optimal or not.

Solution:

$$\begin{array}{rrcrl} z = \max & x_1 & + & 5x_2 & \\ \text{s.t.} & -x_1 & + & 3x_2 & \leq 6 \\ & -4x_1 & - & 4x_2 & \leq -5 \\ & & & x_1 & \leq 2 \\ & & & x_1 & \geq 0 \\ & & & x_2 & \geq 0 \end{array}$$

```
|-----+----+----+----+----+----+----+----|
|     | x1 | x2 | x3 | x4 | x5 | -z |  b |
|-----+----+----+----+----+----+----+----|
| I   | -1 |  3 |  1 |  0 |  0 |  0 |  6 |
| II  | -4 | -4 |  0 |  1 |  0 |  0 | -5 |
| III |  1 |  0 |  0 |  0 |  1 |  0 |  2 |
| IV  |  1 |  5 |  0 |  0 |  0 |  1 |  0 |
|-----+----+----+----+----+----+----+----|
```

The basic solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $x_4 = -5$ and $x_5 = 2$. Its value is 0. The solution is not feasible.

c. Consider the following tableau:

```
|-----+----+----+----+------+----+----+------|
|     | x1 | x2 | x3 | x4   | x5 | -z | b    |
|-----+----+----+----+------+----+----+------|
| I   |  0 |  4 |  1 | -1/4 |  0 |  0 | 29/4 |
| II  |  1 |  1 |  0 | -1/4 |  0 |  0 | 5/4  |
| III |  0 | -1 |  0 | 1/4  |  1 |  0 | 3/4  |
| IV  |  0 |  4 |  0 | 1/4  |  0 |  1 | -5/4 |
|-----+----+----+----+------+----+----+------|
```

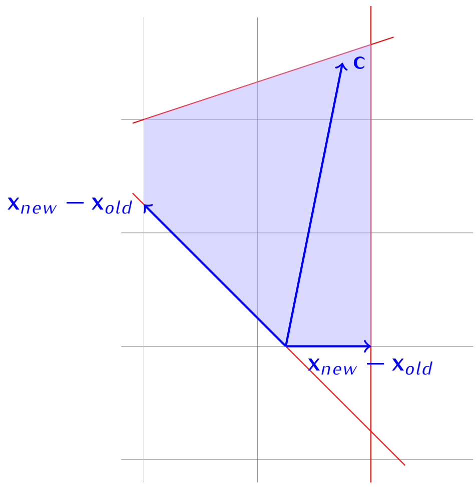
and the following three pivoting rules:

- largest coefficient
- largest increase
- steepest edge.

Which entering and leaving variables would each of them indicate? In this specific case, which rule would be convenient to follow? Report the details of the computations for the first two rules and carry out graphically the application of the third rule using the plot in the figure above (tikz code to reproduce the figure available in the online version.)

Solution:

- The two candidate entering variables are $x2$ and $x4$. The reduced cost of $x2$ is larger hence that is the entering variable. The leaving variable is consequently given by the ratio test and is $x1$ since $5/4 < 29/16$.
- The two candidate entering variables are $x2$ and $x4$. The increase possible with $x2$ is $\min\{29/4 \cdot 1/4, 5/4 \cdot 1\} \cdot 4 = 5$ while the increase with $x4$ is $\min\{3/4 \cdot 4\} \cdot 1/4 = 3/4$. Hence $x2$ is the entering variable and the leaving variable is $x1$.
- In the figure we plot the vector $\mathbf{c}$ which is the perpendicular to the objective function and the two vectors corresponding to the movement we would take by the iteration of the simplex. The angle between $\mathbf{c}$ and $\mathbf{x}_{new} - \mathbf{x}_{old}$ is smaller for the decision x_2 entering x_1 leaving.



None of the three rules is convenient, the best would be to let $x4$ enter and $x5$ leave, we would reach the optimal solution in less iterations.

Exercise 6*

Consider the following problem:

$$\begin{aligned} \max \ & z = 4x_2 \\ \text{s.t.} \ & 2x_2 \geq 0 \\ & -3x_1 + 4x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

a. Write the LP in equational standard form and say why it does not provide immediately an initial feasible basis for the simplex method.

Solution:

In the equational standard form we have a negative b term. The implication of this is that the initial solution of the simplex is infesible because $x_B \not\geq 0$. If we try to make the term positive we end up not having an identity matrix in the tableau.

b. To overcome the situation of infeasible basis construct the auxiliary problem for a phase I-phase II solution approach. Determine which variables are initially in basis and which are not in basis in the auxiliary problem.

c. Answer the following questions

i) Is the initial basis in the auxiliary problem feasible in the original problem?
ii) Is it optimal in the auxiliary problem?
iii) Is it degenerate?
iv) Can we say at this stage if phase I will terminate?
v) If it will terminate, can we say at this stage that it will terminate with a basis that corresponds to a feasible solution in the original problem?
vi) Solve the problem by carrying out Phase I and Phase II of the simplex algorithm.

Exercise 7*

Show that the dual of $\max\{c^T x \mid Ax = b, x \geq 0\}$ is $\min\{y^T b \mid y^T A \geq c\}$.

Solution:
This was shown in class by means of the Lagrangian approach.
Let's show it here by the bounding method.
Given $\max\{c^T x \mid Ax = b, x \geq 0\}$ we search for multipliers $y \in \mathbb{R}^n$ such that $y^T Ax = y^T b$ (since we have equalities, the multipliers can be both positive or negative as we do not need to ensure the maintainance of the direction of the inequality). To ensure that we find an upper bound and hence have $c^T x \leq y^T Ax$, we impose $y^T A \geq c^T$ (since $x \geq 0$). Hence, the best upper bound will be given by solving $\min\{y^T b \mid y^T A \geq c^T\}$ (recalling from linear algebra that $(AB)^T = B^T A^T$, we can rewrite: $\min\{y^T b \mid A^T y \geq c\}$, which is the form we would obtain using the recipe method.)

Exercise 8*

Consider the following LP problem:

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 30 \\ & x_1 + 2x_2 \geq 10 \\ & x_1 - x_2 \leq 1 \\ & x_2 - x_1 \leq 1 \\ & x_1 \geq 0 \end{aligned}$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Solution:
The dual is:

$$\begin{aligned} \max\ & 30y_1 + 10y_2 + y_3 + y_4 \\ & 2y_1 y_2 + y_3 - y_4 \geq 2 \\ & 3y_1 + 2y_2 - y_3 + y_4 = 3 \\ & y_1, y_3, y_4 \geq 0 \\ & y_2 \leq 0 \end{aligned}$$

We use the complementary slackness theorem.

$$\begin{cases} 2y_1 y_2 + y_3 - y_4 = 2 \\ 3y_1 + 2y_2 - y_3 + y_4 = 3 \\ y_2 = 0 \\ y_3 = 0 \end{cases}$$

The first because the corresponding variable of the primal is ¿0, the second for the same reason or however because it is already tight by definition, the third and fourth equation are a consequence of the fact that substituting the value of the primal variables variables in the primal problem, the second and third constraints are binding. What we obtain is a linear system of four equations in four variables that we can solve to find the value of the variables of the dual problem.

Exercise 9*

Consider the problem

$$\begin{array}{ll} \text{maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]