

DM559/DM545 – Linear and integer programming

Sheet 6, Spring 2017 [pdf format]

Solution:
Included.

Exercise 1*

Consider the following problem:

$$\begin{aligned} &\text{maximize} && z = x_1 - x_2 \\ &\text{subject to} && x_1 + x_2 \leq 2 \\ & && 2x_1 + 2x_2 \geq 2 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.

There are two ways to carry out phase I.

- In lecture 4 we saw a way to find an initial feasible basis via an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. Phase I is thus carried out by solving an auxiliary LP problem whose solution gives an initial feasible basis or a proof of infeasibility.
- The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:
 1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
 2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| : a_{i^*j} < 0\}$
 3. Update the tableau around the pivot in the same way as with the primal simplex.
 4. Stop if no right-hand side term is negative.

Opposite to the primal simplex method, the dual simplex method iterates through infeasible basis solutions, while maintaining them optimal, and stops when a feasible solution is reached.

Duality can help us with the issue of initial feasible basis solutions. In the problem above, if the objective function was $w = -x_1 - x_2$, then the initial basis solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. But with objective function z the simplex has infeasible initial basis in both problems. However we can change temporarily the objective function z with w and apply the dual simplex method. When it stops we reached a feasible solution that is optimal with respect to w . We can then reintroduce the original objective function and continue iterating with the primal simplex. This phase I–phase II simplex method is also called the *dual–primal simplex method*. Apply this method to the problem above and verify that it leads to the same solution as in point 1.

Solution:

$$\begin{aligned} \max \quad & x_1 - x_2 = z \\ & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We put in equational standard form by introducing a slack variable $s_1 \geq 0$ and a surplus variable $s_2 \geq 0$:

$$\begin{aligned} \max \quad & x_1 - x_2 = z \\ & x_1 + x_2 + s_1 = 2 \\ & 2x_1 + 2x_2 - s_2 = 2 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

This form is not canonical and therefore the first tableau does not have a feasible starting solution.

Auxiliary Problem Approach

We proceed by

- Phase I solving an auxiliary/augmented problem
- Phase II continuing with ordinary simplex

Phase I We introduce an auxiliary variable $a_1 \geq 0$ in the constraint that makes the infeasibility to yield a canonical form:

$$\begin{aligned} \max \quad & x_1 - x_2 = z \\ & x_1 + x_2 + s_1 = 2 \\ & 2x_1 + 2x_2 - s_2 + a_1 = 2 \\ & x_1, x_2, s_1, s_2, a_1 \geq 0 \end{aligned}$$

Now we have a canonical form

x1	x2	s1	s2	a1	-z	b
1	1	1	0	0	0	2
2	2	0	-1	1	0	2
1	-1	0	0	0	1	0

This problem will have the same solution as the original one only when $a_1 = 0$. We can then solve

- an *augmented problem* by introducing the following objective function $\max w = x_1 - x_2 - Ma_1$, where M is a large enough constant or
- an *auxiliary problem* $\min w = a_1 = -\max(-a_1)$.

Let's take the auxiliary problem, if $w^* > 0$ then we will conclude that the feasibility region of the original problem is empty. Otherwise, if $w^* = 0$, then this implies that $a_1 = 0$ and we found a feasible solution. Let's proceed by setting up the tableau of the auxiliary problem

x1	x2	s1	s2	a1	-z	-w	b
1	1	1	0	0	0	0	2
2	2	0	-1	1	0	0	2
1	-1	0	0	0	1	0	0
0	0	0	0	-1	0	1	0

This is not in canonical form but it is easy to bring it to canonical form: just add the second row to the last one.

	x_1	x_2	s_1	s_2	a_1	$-z$	$-w$	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
	1	-1	0	0	0	1	0	0
	2	2	0	-1	0	0	1	2

The variables s_1, a_1 give us a feasible basis now. It is not optimal. We proceed with the pivot operations. In this case it is worth noting that in the ratio rule, we do not consider the third row since that row corresponds to the original objective function and not to a constraint.

We make x_1 enter the basis and consequently a_1 goes out. The pivot is 2 and the new tableau:

	x_1	x_2	s_1	s_2	a_1	$-z$	$-w$	b
$R_1' = R_1 - R_2'$	0	0	1	1/2	-1/2	0	0	1
$R_2' = R_2/2$	1	1	0	-1/2	1/2	0	0	1
$R_3' = R_3 - R_2'$	0	-2	0	1/2	-1/2	1	0	-1
$R_4' = R_4 - R_2$	0	0	0	0	-1	0	1	0

The tableau is optimal. One non basic variable has reduced cost null, which indicates that there are infinite solutions, but this is not relevant now. The relevant thing is that $w^* = 0$ hence the minimum of the auxiliary problem is 0 and hence there is a feasible solution for $a_1 = 0$. This concludes the Phase I of the algorithm since a feasible solution for the auxiliary problem is feasible also for the original problem.

Phase II We throw away the last row and the second last column from the tableau since we do not need them anymore.

	x_1	x_2	s_1	s_2	a_1	$-z$	b
	0	0	1	1/2	-1/2	0	1
	1	1	0	-1/2	1/2	0	1
	0	-2	0	1/2	-1/2	1	-1

The tableau is not optimal. The basic solution corresponding to this tableau is feasible but not optimal. We bring s_4 in the basis and make s_3 leave. The new tableau is:

	x_1	x_2	s_1	s_2	a_1	$-z$	b
$R_1' = 2 \cdot R_1$	0	0	2	1	-1	0	2
$R_2' = R_2 + R_1$	1	1	0	-1/2	1/2	0	2
$R_3' = R_3 - R_1$	0	-2	-1	0	0	1	-2

The tableau is now optimal. The optimal solution is $x = (2, 0)$ and $z^* = 2$.

Dual-Primal Simplex Method

Phase I Let's write the dual of the problem above:

$$\begin{array}{rcl}
 \max & x_1 - x_2 = z & \\
 & x_1 + x_2 \leq 2 & \\
 & 2x_1 + 2x_2 \geq 2 & \\
 & x_1, x_2 \geq 0 & \\
 \min & 2y'_1 + 2y'_2 = w & \\
 & y'_1 + 2y'_2 \geq 1 & \\
 & y'_1 + 2y'_2 \geq -1 & \\
 & y'_1 \geq 0 & \\
 & y'_2 \leq 0 &
 \end{array}
 \quad
 \begin{array}{l}
 y'_1 = y_1 \\
 y'_2 = -y_2 \rightarrow
 \end{array}$$

$$\begin{array}{rcl}
 \min & 2y_1 - 2y_2 = w & \\
 & y_1 - 2y_2 \geq 1 & \\
 & y_1 - 2y_2 \geq -1 & \\
 & y_1, y_2 \geq 0 &
 \end{array}$$

If we put this LP problem in standard form:

$$\begin{array}{rcl}
 \max & -2y_1 + 2y_2 = w & \\
 & -y_1 + 2y_2 \leq -1 & \\
 & -y_1 + 2y_2 \leq 1 & \\
 & y_1, y_2 \geq 0 &
 \end{array}$$

and looking at the tableau:

y1	y2	s1	s2	-z	b
-1	2	1	0	0	-1
-1	2	0	1	0	1
-2	2	0	0	1	0

we see that the initial tableau like for the primal problem is infeasible.

However, the dual problem has an advantage, if we change temporarily the objective function of the primal problem to $\eta = -x_1 - x_2$, the dual problem becomes:

$$\begin{array}{rcl}
 \max & -x_1 - x_2 = \eta & \\
 & x_1 + x_2 \leq 2 & \\
 & 2x_1 + 2x_2 \geq 2 & \\
 & x_1, x_2 \geq 0 & \\
 \min & 2y_1 - 2y_2 = \gamma & \\
 & y_1 - 2y_2 \geq 1 & \\
 & y_1 - 2y_2 \geq -1 & \\
 & y_1, y_2 \geq 0 & \\
 \max & -2y_1 + 2y_2 = \gamma & \\
 & -y_1 + 2y_2 \leq 1 & \\
 & -y_1 + 2y_2 \leq -1 & \\
 & y_1, y_2 \geq 0 &
 \end{array}$$

and the corresponding tableau has an easy basic feasible solution:

y1	y2	s1	s2	-z	b
-1	2	1	0	0	1
-1	2	0	1	0	1
-2	2	0	0	1	0

We can then solve to optimality with the primal simplex: the variable y_2 enters the basis and the variable s_2 exits. The new tableau becomes:

	y1	y2	s1	s2	-z	b
R1	-1	2	1	0	0	1
R2' = R2/2	-1/2	1	0	1/2	0	1/2

$$\begin{array}{c}
 | R3'=R3-R2 | \quad -1 | \quad 0 | \quad 0 | \quad -1 | \quad 1 | \quad -1/2 | \\
 |-----+-----+-----+-----+-----+-----+-----|
 \end{array}$$

and it is optimal. At this stage we can go back to the primal problem where we now have a feasible solution, change the objective function back to the original one and continue with the primal simplex.

We can do the same iteration on the primal but with the dual simplex. Let's write the tableau of the primal with the objective function temporarily changed and keeping the old objective as well:

$$\begin{array}{c}
 | x1 | x2 | s1 | s2 | -z | -e | b | \\
 |-----+-----+-----+-----+-----+-----+-----| \\
 | 1 | 1 | 1 | 0 | 0 | 0 | 2 | \\
 | -2 | -2 | 0 | 1 | 0 | 0 | -2 | \\
 | 1 | -1 | 0 | 0 | 1 | 0 | 0 | \\
 | -1 | -1 | 0 | 0 | 0 | 1 | 0 | \\
 |-----+-----+-----+-----+-----+-----+-----|
 \end{array}$$

As we see we have the conditions of the dual simplex satisfied, the tableau is optimal but not feasible. Let's make an iteration of the dual simplex. We choose the row with negative b term and the column with negative pivot that minimizes the ratio test: $|c/a|$. We choose the second row and the second column (again watch out that we do not consider the row of the add old objective to decide the row). In other terms we try to make the solution feasible while minimizing the loss in quality. The operations to update the tableau remain the same as for the primal simplex. We obtain:

$$\begin{array}{c}
 | \quad \quad \quad | x1 | x2 | s1 | s2 | -z | -e | b | \\
 |-----+-----+-----+-----+-----+-----+-----| \\
 | R1'=R1-R2' | 0 | 0 | 1 | 1/2 | 0 | 0 | 1 | \\
 | R2'=-1/2R2 | 1 | 1 | 0 | -1/2 | 0 | 0 | 1 | \\
 | R3'=R3+R2' | 2 | 0 | 0 | -1/2 | 1 | 0 | 1 | \\
 | R4'=R4+R2' | 0 | 0 | 0 | -1/2 | 0 | 1 | 1 | \\
 |-----+-----+-----+-----+-----+-----+-----|
 \end{array}$$

This tableau is optimal for the dual simplex, this means that a feasible solution for the primal problem has been found: $(0, 1, 1, 0)$. We can now proceed with the primal simplex.

Note that the considerations on the dual problem made above were just for explanation purposes, when solving our LP problem we do not need to write down the dual form of it or its tableaux. Instead, we just need to switch from dual simplex to primal simplex always working on the original (the primal) formulation of the problem. The dual simplex method simply a new way of picking the entering and leaving variables in a sequence of primal tableaux.

Phase II We can now remove the temporary objective function and the corresponding column and proceed with the primal simplex.

$$\begin{array}{c}
 | x1 | x2 | s1 | s2 | -z | b | \\
 |-----+-----+-----+-----+-----+-----| \\
 | 0 | 0 | 1 | 1/2 | 0 | 1 | \\
 | 1 | 1 | 0 | -1/2 | 0 | 1 | \\
 | 2 | 0 | 0 | -1/2 | 1 | 1 | \\
 |-----+-----+-----+-----+-----+-----|
 \end{array}$$

x_1 enters the basis and x_2 exits. The tableau is updated consequently:

	x_1	x_2	s_1	s_2	$-z$	b
R1' = R1	0	0	1	1/2	0	1
R2' = R2	1	1	0	-1/2	0	1
R3' = R3 - 2 * R2	0	-2	0	1/2	1	-1

A reduced cost is still positive, hence we make s_2 enters in the basis and s_1 leave. This leads to

	x_1	x_2	s_1	s_2	$-z$	b
R1' = 2 * R1	0	0	2	1	0	2
R2' = R2 + R1	1	1	1	0	0	2
R3' = R3 - R1	0	-2	-1	0	1	-2

The tableau is now optimal and the corresponding basic feasible solution is $x = (2, 0)$ and has value $z^* = 2$.

We can visualize the problem using the LP Grapher tool linked from the course webpage:

Enter the linear programming problem here:

Maximize $z = x - y$ subject to the constraints:

Minimize

Show only the region defined by the following constraints:

$$x + y \leq 2$$

$$2x + 2y \geq 2$$

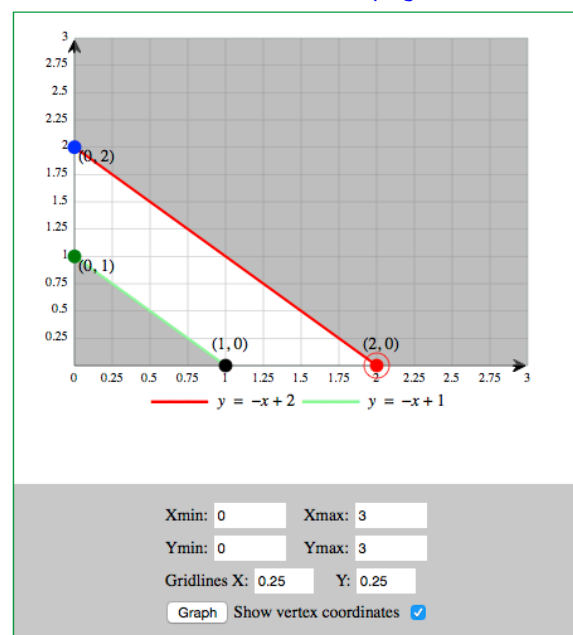
LP Examples Graphing Examples Solve

Rounding: 4 decimal places Fraction Mode

Erase Everything

The solution will appear below.

Vertex	Lines through vertex	Value of objective
• (0, 2)	$x + y = 2$ $x = 0$	-2
• (2, 0)	$x + y = 2$ $y = 0$	2 Maximum
• (0, 1)	$2x + 2y = 2$ $x = 0$	-1
• (1, 0)	$2x + 2y = 2$ $y = 0$	1



Exercise 2* Sensitivity Analysis and Revised Simplex

A furniture-manufacturing company can produce four types of product using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.
- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.

- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve the following LP problem

$$\begin{aligned} \max \quad & 19x_1 + 13x_2 + 12x_3 + 17x_4 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 \\ & x_1 + x_2 + x_3 + x_4 \leq 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The final tableau has x_1, x_3 and x_4 in basis. With the help of a computational environment such as Python for carrying out linear algebra operations, address the following points:

- a) Write $A_B, A_N, A_B^{-1}A_N$, the final simplex tableau and verify that the solution is indeed optimal.

Solution:

The initial tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	b
3	2	1	2	1	0	0	0	225
1	1	1	1	0	1	0	0	117
4	3	3	4	0	0	1	0	420
19	13	12	17	0	0	0	1	0

We know that there will be 3 variables in basis. The text of the problem tells us which these 3 variables are: 1, 3, 4. Hence,

$$A_B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix} \quad A_N = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

We can calculate $A_B^{-1}A_N$ in Python or in R:

```
> B=matrix(c(3,1,2,1,1,1,4,3,4),byrow=TRUE,ncol=3)
> B1=solve(B)
> B1*B1 # check to make sure it is correct!
      [,1] [,2] [,3]
[1,]  1   0   0
[2,]  0   1   0
[3,]  0   0   1
> N=matrix(c(2,1,0,0,1,0,1,0,3,0,0,1),ncol=4,byrow=TRUE)
> B1*N
      [,1] [,2] [,3] [,4]
[1,]  1   1   2  -1
[2,]  1   0   4  -1
[3,] -1  -1  -5   2
> cN=c(13,0,0,0)
> cB=c(19,12,17)
> cN-cB*B1*N
      [,1] [,2] [,3] [,4]
```

```
[1,] -1 -2 -1 -3
> cB%*%B1%*%c(225,117,420)
> 1827
```

This code gives us:

$$\bar{A} = A_B^{-1}A_N = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 4 & -1 \\ -1 & -1 & -5 & 2 \end{bmatrix} \quad x_B^* = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = A_B^{-1}b = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix}$$

$$\bar{c}_N = [\bar{c}_2 \ \bar{c}_5 \ \bar{c}_6 \ \bar{c}_7] = [-1 \ -2 \ -1 \ -3]$$

and we can write the final tableau as:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	-z	b
1	1	0	0	1	2	-1	0	39
0	1	1	0	0	4	-1	0	48
0	-1	0	1	-1	-5	2	0	30
0	-1	0	0	-2	-1	-3	1	-1827

Since all reduced costs are negative then the tableau and the corresponding solution are optimal.

- b) What is the increase in price (reduced cost) that would make product x_2 worth to be produced?

Solution:

The increase in price of a quantity strictly larger than 1 would make the product 2 worth being produced. Indeed, let $c'_2 = c_2 + \delta$ be the new price. We know that the coefficient in the objective function goes in the reduced cost calculation multiplied by 1. Hence, to have a positive reduced cost we have:

$$-1 + \delta > 0 \quad \implies \quad \delta > 1$$

We could also recalculate the reduced cost from scratch using the multipliers π : $c'_2 + \sum_{i=1}^3 \pi_i a_{i2}$. The value of π_i are read from the final tableau and they correspond to the reduced costs of the slack variables, ie, $(-2, -1, -3)$.

- c) What is the marginal value (shadow price) of an extra hour of work or amount of metal and wood?

Solution:

The marginal values are the values of the dual variable y_1, y_2, y_3 . From the strong duality theorem, we know that $y_i = -\pi_i = -\bar{c}_{n+i}$, $i = 1..m$. Hence, $\mathbf{y} = (2, 1, 3)$.

An extra hour of work has marginal value of 2, that is, having one unit more of work would improve the revenue by 2. For the other two resources the marginal values are 1 and 3, respectively.

We can cross check these conclusions: by the complementary slackness theorem, the fact that all three dual variables are strictly positive indicates that all three constraints in the primal are active \equiv tight \equiv binding. Hence, it makes sense to have that an increase in the capacity of those constraints implies an increase in the profit. The conclusion that all three constraints are tight can be also reached by the fact that the slack variables are 0 in the final tableau. If some constraint was not tight, then the marginal value of the corresponding resource would be zero since an increase in its capacity does not imply an immediate improvement in total profit.

- d) Are all resources totally utilized, i.e. are all constraints “binding”, or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

Solution:

Since all dual variables are strictly larger than zero, then all constraints are binding. Indeed for the complementary slackness theorem, we have that:

$$\left(b_i - \sum_{j=1}^n a_{ij}x_j^* \right) y_i^* = 0, \quad i = 1, \dots, m$$

- e) From the economical interpretation of the dual why product x_2 is not worth producing? What is its imputed cost?

Solution:

It is not worth producing 2 because $\sum_i y_i a_{i2} > c_2$, that is, we are better off selling the raw materials to produce the product. Indeed y_i is the price of one unit of resource i and a_{i2} is the amount of i necessary to produce 2.

$$\sum_i y_i a_{i2} = 2 * (2) + 1 * (1) + 3 * (3) = 14 > 13$$

Solve the following variations:

1. The net profit brought in by each desk increases from 13 Euro to 15 Euro.

Solution:

We saw earlier that if the price of product 2 increases by more than 1 then the reduced cost becomes positive and it enters the basis. We can iterate the revised simplex as follows:

Step 1 and 2 to determine the entering variable are already done in the point a) above.

We need to do Step 3 to determine the leaving variable: we need to find the constraint that limit the increase of x_2 , θ . We solve first $A_B d = a$ in d . Here, a is the column of the matrix A (augmented with the slack variables) from the initial tableau corresponding to the entering variable x_2 . We use the inverse of A_B calculated earlier in a) above in R:

$$\begin{array}{l} > B^{-1} * c(2,1,3) \\ \quad [, 1] \\ [1,] \quad 1 \\ [2,] \quad 1 \\ [3,] \quad -1 \end{array}$$

that is

$$d = A_B^{-1} a = A_B^{-1} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then the new solution x_B is derived from the old one by means of d and the increase θ :

$$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \theta \geq 0$$

The increase θ must be such that the value of the variables still remains feasible, ie, $x_i \geq 0$. Hence $\theta \leq 39$ and the leaving variable is x_1 , since it is the one that goes to zero. The new solutions is

$$x_B = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \theta = \begin{bmatrix} 39 - 39 \\ 48 - 39 \\ 30 + 39 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 69 \end{bmatrix}$$

and the objective value:

```
> c=c(19,15,12,17)
> c%*%c(0,39,9,69)
      [,1]
[1,] 1866
```

2. The availability of metal increases from 117 to 125 units per day

Solution:

This is a change in the RHS term of constraint 2. The optimality of the current solution does not change, since all reduced costs stay negative, but we need to check if we are still feasible. We need to look at the final tablea and recompute the b of all constraints. We can do this with $A_B^{-1}b$:

```
> b=c(225,125,420)
> B1%*%b
      [,1]
[1,]   55
[2,]   80
[3,]  -10
```

The last cosntraint becomes negative, hence we need to iterate with the dual simplex.

3. The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood and bring in a net profit of 14 Euro.

Solution:

We need to check if the reduced cost of the new variable would become positive by computing $c_0 + \sum_i \pi_i a_{ij}$:

```
> 14-3*2-1*1-2*3
[1] 1
```

which is positive, hence we need to iterate as done in point 1).

4. The number of chairs produced must be at most five times the numbers of desks

Solution:

This corresponds to introduce a new constraint: $x_3 \leq 5x_2$. In the new standard form we have a new slack variable x_8 . Adding the constraint in the tableau and bringing back the tableau in canonical standard form we observe that a RHS term becomes negative. Hence, we need to iterate with the dual simplex. After on iteration with the dual simplex, the final tableau becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 4/3 & -5/6 & 1/6 & 0 & 31 \\ 0 & 1 & 0 & 0 & 0 & 2/3 & -1/6 & -1/6 & 0 & 8 \\ 0 & 0 & 0 & 1 & -1 & -13/3 & 11/6 & -1/6 & 0 & 38 \\ 0 & 0 & 1 & 0 & 0 & 10/3 & -5/6 & 1/6 & 0 & 40 \\ 0 & 0 & 0 & 0 & -2 & -1/3 & -19/6 & -1/6 & 1 & -1819 \end{bmatrix}$$

If after the introduction of the constraint the current solution had stayed feasible then we would have needed to check whether its was also optimal. We can either repeat the steps done at part 1 above to compute the new reduced costs or we can include the new row in the final tableau and proceed to put the tableau in canonical form. Then we look at the value of the reduced costs.

Exercise 3

Solve the systems $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1 \ 2 \ 3]$ and $E_1 E_2 E_3 E_4 \mathbf{d} = [1 \ 2 \ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution:

This exercise is to show that the two systems can be solved quite easily. Let's take first $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1 \ 2 \ 3]$, we use the backward transformation and solve the sequence of linear systems:

$$\mathbf{u}^T E_4 = [1 \ 2 \ 3], \quad \mathbf{v}^T E_3 = \mathbf{u}^T, \quad \mathbf{w}^T E_2 = \mathbf{v}^T, \quad \mathbf{y}^T E_1 = \mathbf{w}^T$$

$$\mathbf{u}^T \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [1, 2, 3]$$

Since the eta matrices have always one 1 in two columns then the solution can be read up easily. From the third column we find $u_3 = 3$. From the second column, we find $u_2 = 2$. Substituting in the first column, we find $-0.5u_1 + 3 * 2 + 1 * 3 = 1$, which yields $u_1 = 18$. The next system is:

$$\mathbf{v}^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [18, 2, 3]$$

From the first column we get $v_1 = 18$, from the second column $v_2 = 2$ from the last column $v_3 = 3/24$. The next:

$$\mathbf{w} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = [18, 2, 3/24]$$

...

Exercise 4* Quizzes

Basic Geometric Facts

1. In 4D, how many hyperplanes need to intersect to give a point?

Solution:

4

2. In 4D, can a point be described by more than 4 hyperplanes?

Solution:

Yes, just think of a pyramid in 3D

3. Consider the intersection of n hyperplanes in n dimensions: when does it uniquely identify a point?

Solution:

when the rank of the matrix A of the linear system is n (or A is nonsingular)

Vertices of Polyhedra:

Consider the polyhedron described by $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, that is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

4. How many constraints are *active* in a *vertex* of a polyhedron $Ax \leq \mathbf{b}$, $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$?

Solution:

at least n , rank of matrix of active constraints is n

5. Does every point x that activates n constraints form a vertex of the polyhedron?

Solution:

no, some may be not feasible, ie, intersection in a point outside of the polyhedron

6. Can a vertex activate more than n constraints?

Solution:

Yes, just look at the pyramid in 3 dim. However, the rank of the matrix of active constraints is still n

7. What if there are more variables than constraints? If $m > n$ then we can find a subset and then activate but what if $m < n$, can we have a vertex?

Solution:

No. In LP we deal with this issue by adding slack variables, they make us choose arbitrarily a vertex

8. Combinatorial explosion of vertices: how many constraints and vertices has an n -dimensional hypercube?

Solution:

To define a cube we need 6 constraints and there are 2^3 vertices. For an n -hypercube we need $2n$ constraints and there are 2^n vertices

9. If there are m constraints and n variables, $m > n$, what is an upper bound to the number of vertices?

Solution:

the number of possible active constraints is $\binom{m}{n}$ it is an upper bound because:

- some combinations of constraints will not define a vertex, ie, if rows of matrix not independent
- some vertices are outside the polyhedron
- some vertices may activate more than n constraints and hence the same vertex can be given by more than n constraints

Tableaux and Vertices

10. For each of these three statements, say if they are true or false:

- One tableau \implies one vertex of the feasible region
- One tableau \longleftarrow one vertex of the feasible region
- One tableau \iff one vertex of the feasible region

Solution:

One tableau \Leftarrow one vertex of the feasible region degenerate vertices have several tableau associated

11. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rcl} \max & 6x_1 + 8x_2 & \\ & 5x_1 + 10x_2 \leq 60 & \\ & 4x_1 + 4x_2 \leq 40 & \\ & x_1, x_2 \geq 0 & \end{array} \quad \begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & -z & b \\ \hline x_2 & 0 & 1 & 1/5 & -1/4 & 0 & 2 \\ x_1 & 1 & 0 & -1/5 & 1/2 & 0 & 8 \\ \hline & 0 & 0 & -2/5 & -1 & 1 & -64 \end{array}$$

- How many variables (original and slack) can be different from zero?

Solution:

at most 2

- $(x_3, x_4) = (0, 0)$ are non basic, what does this tell us about the constraints?

Solution:

They are active because their dual values are not zero

Let's generalize the previous case. Consider an LP with m constraints, n original variables and m slack variables. In an optimal solution:

- is $m > n$, how many variables (original and slack) can be nonzero at most?

Solution:

at most m

- If $m < n$ how many original variables must be zero at least? In other terms, in a mix planning problem with n products and m , $m < n$ resources, how many products at most will be to be produced in an optimal solution?

Solution:

$n - m$, and hence at most $m < n$ products

Solution:

at most m

12. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rcl} \max & 6x_1 + 8x_2 & \\ & 5x_1 + 10x_2 \leq 60 & \\ & 4x_1 + 4x_2 \leq 40 & \\ & x_1, x_2 \geq 0 & \end{array} \quad \begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & -z & b \\ \hline x_3 & 0 & 0 & 1 & 1/2 & 0 & 1 \\ x_1 & 1 & 1 & 0 & -1/2 & 0 & 1 \\ \hline & 0 & -2 & 0 & 1/2 & 1 & -1 \end{array}$$

- $(x_2, x_4) = (0, 0)$ is non basic, what does this tell us about the constraints?

Solution:

The second constraint is active because its slack x_4 is zero. $x_2 = 0 \implies x_2 \geq 0$ is active.

13. If in the original space of the problem we had 3 variables, and there are 6 constraints, how many constraints would be active?

Solution:

3 constraints. With slack variables we would have 6 variables in all, if any of them is positive the constraint $x_i \geq 0$ of the original variables would be active, otherwise the corresponding constraints of the original problem are active.

14. For the general case with n original variables:
One basic feasible solution \iff a matrix of active constraints has rank n . True or False?

Solution:

True

15. Consider an LP problem with m constraints and n original variables, $m > n$. We saw that in \mathbb{R}^n a point is the intersection of at least n hyperplanes. In LP this corresponds to say that in a vertex there are n active constraints. Let a tableau be associated with a solution that makes exactly $n + 1$ constraints active, what can we say about the corresponding basic and non-basic variable values?

Solution:

one basic variable is zero. Indeed, in the simplex we will have $m + n$ variables and m variables in basis. We saw that the n non basic variables are set to zero and that there is an active constraint for each of them. Hence, if there are $n + 1$ active constraints, there must be another variable that is set to zero. It must be a basic variable.

16. What is the algebraic definition of adjacency in 2, 3 and n dimensions?

Solution:

two vertices are adjacent iff:

- they have at least $n - 1$ active constraints in common
- rank of common active constraints is $n - 1$

17. How does this condition translate in terms of tableau?

Solution:

For what seen above this translates in $n - 1$ variables in common in the tableau