DM559/DM545

Linear and Integer Programming

Introduction to Linear Programming
Notation and Modeling

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Who is here?

DM559 (7.5 ECTS)
59 officially registered
48 handed in 0.1

- Computer Science
  (2nd year, 4th semester)

Prerequisites

- Programming

DM545 (5 ECTS)
26 officially registered

- Math-economy
  (3rd year ?)
- Applied Mathematics
  (2nd year ?)
- Others?

Prerequisites

- Programming
- Linear Algebra (MM505)

Outline

Aims of the course

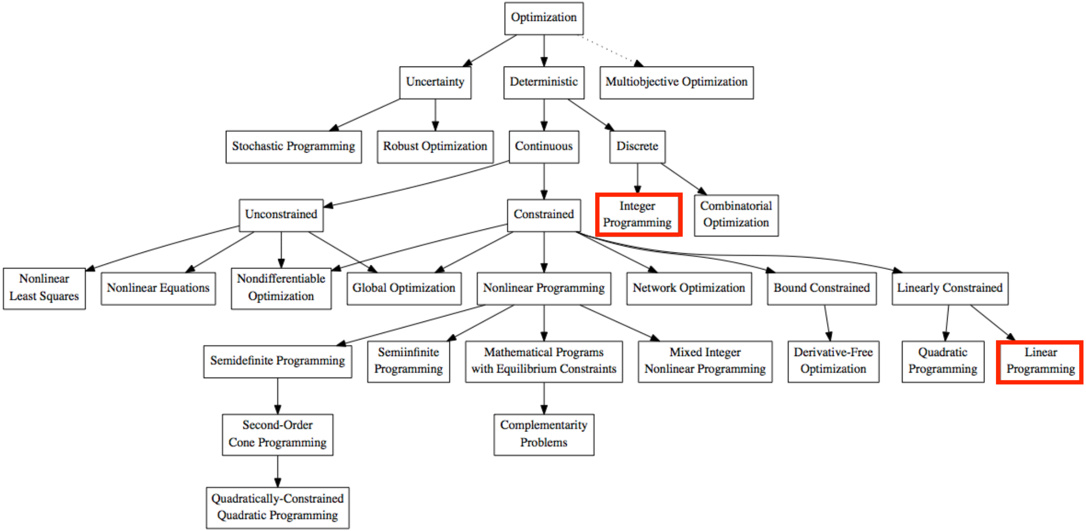
Learn about mathematical optimization:

- linear programming (continuous linear optimization)
- integer programming (discrete linear optimization)

$\rightsquigarrow$ You will see the theory and apply the tools learned to solve real life problems using computer software

Optimization Taxonomy

(NEOS Server, University of Wisconsin)



Contents of the Course

(see Syllabus)

Linear Programming

1 Introduction - Linear Programming, Notation & Modeling
2 Linear Programming, Simplex Method
3 Exception Handling
4 Duality Theory
5 Sensitivity
6 Revised Simplex Method

Integer Linear Programming

7 Modeling Examples, Good Formulations, Relaxations
8 Well Solved Problems
9 Network Optimization Models (Max Flow, Min cost flow, Matching)
10 Cutting Planes & Branch and Bound
11 More on Modeling

Practical Information

Teacher: Marco Chiarandini (www.imada.sdu.dk/~marco/)
Instructor: Kristoffer Abell
Sections (hold): H1, H2, M1

Alternative views of the schedule:

- mitsdu.sdu.dk, SDU Mobile
- Official course description (læserplanen)
- http://www.imada.sdu.dk/~marco/DM545

Schedule:

- Introductory classes: $\sim$ 28 hours ($\sim$ 14 classes)
- Training classes: $\sim$ 24 hours ($\sim$ 12 classes)
  - Exercises: 20 hours
  - Laboratory: 4 hours (2 classes)

Communication Means

- BlackBoard (BB) ⇔ Main Web Page (WP)
  (link `http://www.imada.sdu.dk/~marco/DM545`)
- Announcements in BlackBoard
- Write to Marco (`marco@imada.sdu.dk`) and to instructor
- Ask peers
- You are welcome to visit me in my office in working hours (8-16)

⇝ It is good to ask questions!!

⇝ Let me know if you think we should do things differently!

Sources

Linear Programming:

LN Lecture Notes (continously updated)

MG J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

Integer Programming:

LN Lecture Notes (continously updated)

Wo L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

Other books and articles:

HL Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

... see webpage

Course Material

Main Web Page (WP) is the main reference for list of contents (ie[1], syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- resources for programming tasks

[1] ie = id est, eg = exempli gratia, wrt = with respect to

Assessment

Accomplishment of the following is required for 5 ECTS:

- Two obligatory Assignments, pass/fail, evaluation by teacher
    - modeling + describing + programming in Python with Gurobi
    - (language: Danish and/or English)
    - individual
- 4 hour written exam, 7-grade scale, external censor
    - similar to exercises in class and past exams
    - on June 4

Training Sessions

- Prepare the starred exercises in advance to get out the most
- Try the others after the session
- Best if carried out in small groups
- Exercises are examples of exam questions (but not only!)

Concepts from Linear Algebra

Linear Algebra:
manipulation of matrices and vectors with some theoretical background

Linear Algebra

- Matrices and vectors - Matrix algebra
- Inner (dot) product
- Geometric insight
- Systems of Linear Equations - Row echelon form, Gaussian elimination
- Matrix inversion and determinants
- Rank and linear dependency

DM545 has an (obligatory) assignment on this (sheet0).

Coding

- gives you the ability to create new and useful artifacts with just your mind and your fingers,
- allows you to have more control of your world as more and more of it becomes digital,
- is just fun.

It can also help you understand math.
Beside:

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand

You can learn by doing, interacting with Python.

from Coding the Matrix by Philip Klein

- Python 3.6 (or python 2.7 with `import from __future__`) + Gurobi (100 000 Dkk) – (note: gurobipy does not work with Python 3 to 3.5) See links for installation from Tools section at course webpage
- ipython, jupyter, jupyterLab (= interactive python)? Or Spyder3 or Atom.

Outline

What is Operations Research?

Operations Research (aka, Management Science, Analytics):
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system,
usually under conditions requiring the allocation of scarce resources,
by means of **mathematics** and **computer science**.

Quantitative methods for planning and analysis.

It encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency:

- simulation,
- mathematical optimization,
- queueing theory and other stochastic-process models,
- Markov decision processes
- econometric methods,
- data envelopment analysis,
- neural networks,
- expert systems

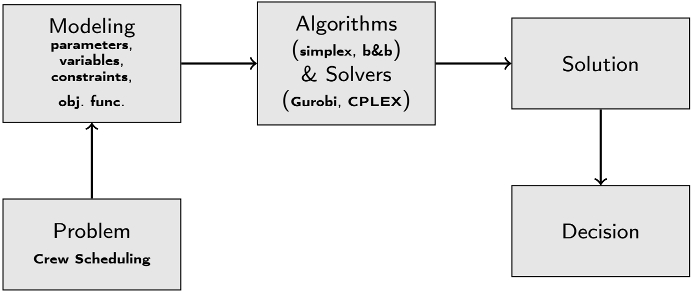
Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
  - Cheapest
  - Shortest route
  - Fewest number of people
- Not all plans are feasible - there are constraining rules
  - Limited amount of available resources
- It can be extremely difficult to figure out what to do

OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Central Idea

Build a mathematical model describing exactly what one wants, and what the "rules of the game" are. However, **what is a mathematical model and how?**

Mathematical Modeling

- Find out exactly what the decision maker needs to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Decision Variables of suitable type (continuous, integer, binary) corresponding to the needs
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

Outline

Resource Allocation

In manufacturing industry, factory planning: find the best product mix.

Example

A factory makes two products standard and deluxe.

A unit of standard gives a profit of 6k Dkk.
A unit of deluxe gives a profit of 8k Dkk.

The grinding and polishing times in terms of hours per week for a unit of each type of product are given below:

	Standard	Deluxe
(Machine 1) Grinding	5	10
(Machine 2) Polishing	4	4

Grinding capacity: 60 hours per week
Polishing capacity: 40 hours per week
Q: How much of each product, standard and deluxe, should we produce to maximize the profit?

Mathematical Model

Decision Variables

$x_1 \geq 0$ units of product standard
$x_2 \geq 0$ units of product deluxe

Object Function

$\max\ 6x_1 + 8x_2$ maximize profit

Constraints

$$
\begin{aligned}
5x_1 &+ 10x_2 &\leq 60 \quad &\text{Grinding capacity} \\
4x_1 &+ 4x_2 &\leq 40 \quad &\text{Polishing capacity}
\end{aligned}
$$

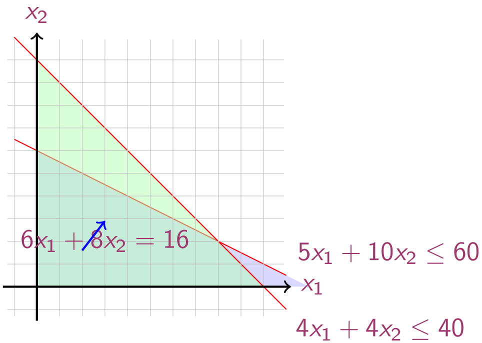
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1 & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$j = 1, 2, \dots, n$	products
$i = 1, 2, \dots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}$	profit per unit of product j
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
& \dots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & \mathbf{c}^T\mathbf{x} \\
 & A\mathbf{x} & \leq & \mathbf{b} \\
 & \mathbf{x} & \geq & 0
\end{array}
$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m \\ & x_j \ge 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max \quad 6x_1 &+ 8x_2 \\ 5x_1 &+ 10x_2 \le 60 \\ 4x_1 &+ 4x_2 \le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ A\mathbf{x} &\le \mathbf{b} \\ \mathbf{x} &\ge 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \ge 0$$

Outline

Duality

Resource Valuation problem: Determine the value of the raw materials on hand such that:
(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

z_i	value of a unit of raw material i
$\sum_{i=1}^m b_i z_i$	total expenses for buying or opportunity cost (cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal: for the outside company to minimize the total expenses;
for the owing company to minimize the lost opportunity cost

$$\min \sum_{i=1}^{m} b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} z_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\min \sum_{i=1}^{m} y_i b_i + \cancel{\sum_i \rho_i b_i}$$

$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1 \ldots n$$

$$y_i \geq 0, \quad i = 1 \ldots m$$

Dual Problem

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m$$

$$x_j \geq 0, \quad j = 1, \ldots, n$$

Primal Problem