DM545

Linear and Integer Programming

Lecture 12

Cutting Planes

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

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Valid Inequalities

- IP: $z = \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\}, X = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_+\}$
- Proposition: $\text{conv}(X) = \{\mathbf{x} : \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}, \mathbf{x} \geq 0\}$ is a polyhedron
- LP: $z = \max\{\mathbf{c}^T\mathbf{x} : \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}, \mathbf{x} \geq \mathbf{0}\}$ would be the best formulation
- Key idea: try to approximate the best formulation.

Definition (Valid inequalities)

$\mathbf{ax} \leq \mathbf{b}$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\mathbf{ax} \leq \mathbf{b}\ \forall \mathbf{x} \in X$

Which are useful inequalities? and how can we find them?
How can we use them?

Example: Pre-processing

- $X = \{(x, y) : x \leq 999y; 0 \leq x \leq 5, y \in \mathbb{B}^1\}$

$$x \leq 5y$$

- $X = \{x \in \mathbb{Z}^n_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6 + \frac{6}{11}$$
$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

- Capacitated facility location:

$$\sum_{i \in M} x_{ij} \leq b_j y_j \quad \forall j \in N \qquad\qquad x_{ij} \leq b_j y_j$$
$$\sum_{j \in N} x_{ij} = a_i \quad \forall i \in M \qquad\qquad x_{ij} \leq a_i$$
$$x_{ij} \geq 0, y_j \in B^n \qquad\qquad x_{ij} \leq \min\{a_i, b_j\} y_j$$

Chvátal-Gomory cuts

- $X \in P \cap \mathbb{Z}^n_+, \quad P = \{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} \leq \mathbf{b}\}, \quad A \in \mathbb{R}^{m \times n}$
- $\mathbf{u} \in \mathbb{R}^m_+$, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n\}$ columns of A

CG procedure to construct valid inequalities

1) $\displaystyle\sum_{j=1}^{n} \mathbf{u}\mathbf{a}_j x_j \leq \mathbf{u}\mathbf{b}$ valid: $\mathbf{u} \geq \mathbf{0}$

2) $\displaystyle\sum_{j=1}^{n} \lfloor \mathbf{u}\mathbf{a}_j \rfloor x_j \leq \mathbf{u}\mathbf{b}$ valid: $\mathbf{x} \geq \mathbf{0}$ and $\sum \lfloor \mathbf{u}\mathbf{a}_j \rfloor x_j \leq \sum \mathbf{u}\mathbf{a}_j x_j$

3) $\displaystyle\sum_{j=1}^{n} \lfloor \mathbf{u}\mathbf{a}_j \rfloor x_j \leq \lfloor \mathbf{u}\mathbf{b} \rfloor$ valid for X since $\mathbf{x} \in \mathbb{Z}^n$

Theorem

by applying this CG procedure a finite number of times every valid inequality for X can be obtained

Cutting Plane Algorithms

- $X \in P \cap \mathbb{Z}^n_+$
- a family of valid inequalities $\mathcal{F} : \mathbf{a}^T\mathbf{x} \leq b, (\mathbf{a}, b) \in \mathcal{F}$ for X
- we do not find them all a priori, only interested in those close to optimum

Cutting Plane Algorithm

Init.: $t = 0, P^0 = P$

Iter. t: Solve $\bar{z}^t = \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P^t\}$
let $\mathbf{x}^t$ be an optimal solution
if $\mathbf{x}^t \in \mathbb{Z}^n$ stop, $\mathbf{x}^t$ is opt to the IP
if $\mathbf{x}^t \notin \mathbb{Z}^n$ solve separation problem for $\mathbf{x}^t$ and $\mathcal{F}$
if $(\mathbf{a}^t, b^t)$ is found with $\mathbf{a}^t\mathbf{x}^t > b^t$ that cuts off x^t

$$P^{t+1} = P \cap \{\mathbf{x} : \mathbf{a}^i\mathbf{x} \leq b^i, i = 1, \ldots, t\}$$

else stop (P^t is in any case an improved formulation)

Gomory's fractional cutting plane algorithm

Cutting plane algorithm + Chvátal-Gomory cuts

- $\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{Z}^n\}$
- Solve LPR to optimality

$$\left[\begin{array}{c:c:c:c} I & \bar{A}_N = A_B^{-1}A_N & 0 & \bar{b} \\ \hdashline \bar{c}_B & \bar{c}_N(\leq 0) & 1 & -\bar{d} \end{array}\right]$$

$$x_u = \bar{b}_u - \sum_{j\in N} \bar{a}_{uj}x_j, \quad u \in B$$
$$z = \bar{d} + \sum_{j\in N} \bar{c}_j x_j$$

- If basic optimal solution to LPR is not integer then $\exists$ some row u: $\bar{b}_u \notin \mathbb{Z}^1$.
  The Chvatál-Gomory cut applied to this row is:

$$x_{B_u} + \sum_{j\in N} \lfloor \bar{a}_{uj} \rfloor x_j \leq \lfloor \bar{b}_u \rfloor$$

(B_u is the index in the basis B corresponding to the row u) (cntd)

- Eliminating $x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j$ in the CG cut we obtain:

$$\sum_{j \in N} (\underbrace{\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor}_{0 \le f_{uj} < 1}) x_j \ge \underbrace{\bar{b}_u - \lfloor \bar{b}_u \rfloor}_{0 < f_u < 1}$$

$$\sum_{j \in N} f_{uj} x_j \ge f_u$$

  $f_u > 0$ or else u would not be row of fractional solution. It implies that x^* in which $x^*_N = 0$ is cut out!

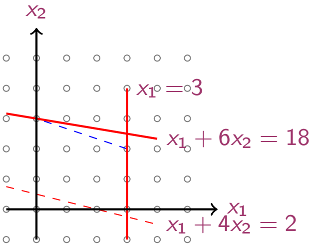
- Moreover: when x is integer, since all coefficients in the CG cut are integer the slack variable of the cut is also integer:

$$s = -f_u + \sum_{j \in N} f_{uj} x_j$$

(theoretically it terminates after a finite number of iterations, but in practice not successful.)

Example

$$
\begin{aligned}
\max\ & x_1 + 4x_2 \\
& x_1 + 6x_2 \leq 18 \\
& x_1 \leq 3 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \text{integer}
\end{aligned}
$$




```
|   | x1 | x2 | x3 | x4 | -z |  b |
|---+----+----+----+----+----+----|
|   |  1 |  6 |  1 |  0 |  0 | 18 |
|   |  1 |  0 |  0 |  1 |  0 |  3 |
|---+----+----+----+----+----+----|
|   |  1 |  4 |  0 |  0 |  1 |  0 |
```

```
|   | x1 | x2 | x3   | x4   | -z |    b |
|---+----+----+------+------+----+------|
|   |  0 |  1 | 1/6  | -1/6 |  0 | 15/6 |
|   |  1 |  0 | 0    | 1    |  0 |    3 |
|---+----+----+------+------+----+------|
|   |  0 |  0 | -2/3 | -1/3 |  1 |  -13 |
```

$x_2 = 5/2, x_1 = 3$

Optimum, not integer

- We take the first row: | | 0 | 1 | 1/6 | -1/6 | 0 | 15/6 |
- CG cut $\sum_{j \in N} f_{uj} x_j \geq f_u \rightsquigarrow \frac{1}{6}x_3 + \frac{5}{6}x_4 \geq \frac{1}{2}$
- Let's see that it leaves out x^*: from the CG proof:

$$\begin{array}{l}
1/6\,(x_1 + 6x_2 \leq 18) \\
5/6\,(x_1 \qquad\;\; \leq 3) \\
\hline
\quad x_1 + x_2 \leq 3 + 5/2 = 5.5
\end{array}$$

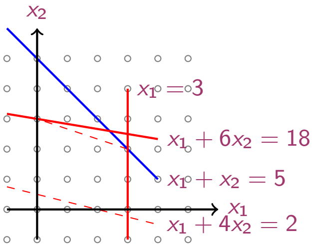
  since x_1, x_2 are integer $x_1 + x_2 \leq 5$

- Let's see how it looks in the space of the original variables: from the first tableau:

$$x_3 = 18 - 6x_2 - x_1$$
$$x_4 = 3 - x_1$$

$$\frac{1}{6}(18 - 6x_2 - x_1) + \frac{5}{6}(3 - x_1) \geq \frac{1}{2} \qquad \rightsquigarrow \qquad x_1 + x_2 \leq 5$$

- Graphically:



- Let's continue:

	x1	x2	x3	x4	x5	-z	b
	0	0	-1/6	-5/6	1	0	-1/2
	0	1	1/6	-1/6	0	0	5/2
	1	0	0	1	0	0	3
	0	0	-2/3	-1/3	0	1	-13

We need to apply dual-simplex (will always be the case, why?)

ratio rule: $\min\{|\frac{c_j}{a_{ij}}| \ : \ a_{ij} < 0\}$

- After the dual simplex iteration:

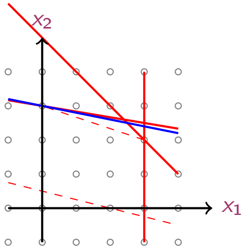
	x1	x2	x3	x4	x5	-z	b
	0	0	1/5	1	-6/5	0	3/5
	0	1	1/5	0	-1/5	0	13/5
	1	0	-1/5	0	6/5	0	12/5
	0	0	-3/5	0	-2/5	1	-64/5

We can choose any of the three rows.

Let's take the third: CG cut:
$\frac{4}{5}x_3 + \frac{1}{5}x_5 \geq \frac{2}{5}$

- In the space of the original variables:

$$4(18 - x_1 - 6x_2) + (5 - x_1 - x_2) \geq 2$$
$$x_1 + 5x_2 \leq 15$$



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