DM559/DM545
Linear and Integer Programming

Linear Programming

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Outline

Outline

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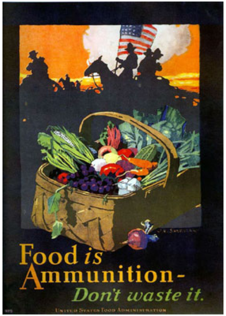
The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
  eat enough but not too much of Vitamin A
  eat enough but not too much of Sodium
  eat enough but not too much of Calories
  ...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread, and
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5,000 and 50,000)

Food	Cost per serving	Vitamin A	Calories
Corn	\$0.18	107	72
2% Milk	\$0.23	500	121
Wheat Bread	\$0.05	0	65

The Mathematical Model

Parameters (given data)

F = set of foods
N = set of nutrients

a_{ij} = amount of nutrient i in food j, $\forall i \in N$, $\forall j \in F$
c_j = cost per serving of food $j, \forall j \in F$
$F_{min,j}$ = minimum number of required servings of food $j, \forall j \in F$
$F_{max,j}$ = maximum allowable number of servings of food $j, \forall j \in F$
$N_{min,i}$ = minimum required level of nutrient $i, \forall i \in N$
$N_{max,i}$ = maximum allowable level of nutrient $i, \forall i \in N$

Decision Variables

x_j = number of servings of food i to purchase/consume, $\forall j \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{j \in F} c_j x_j$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{j \in F} a_{ij} x_j \geq N_{min,i}, \qquad \forall i \in N$$

Constraint Set 2: For each nutrient $i \in N$, do not exceed the maximum allowable level.

$$\sum_{j \in F} a_{ij} x_j \leq N_{max,i}, \qquad \forall i \in N$$

Constraint Set 3: For each food $j \in F$, select at least the minimum required number of servings

$$x_j \geq F_{min,j}, \qquad \forall j \in F$$

Constraint Set 4: For each food $j \in F$, do not exceed the maximum allowable number of servings.

$$x_j \leq F_{max,j}, \qquad \forall j \in F$$

The Mathematical Model

system of equalities and inequalities

$$\min \quad \sum_{j \in F} c_j x_j$$

$$\sum_{j \in F} a_{ij} x_j \geq N_{min,i}, \qquad \forall i \in N$$

$$\sum_{j \in F} a_{ij} x_j \leq N_{max,i}, \qquad \forall i \in N$$

$$x_j \geq F_{min,j}, \qquad \forall j \in F$$

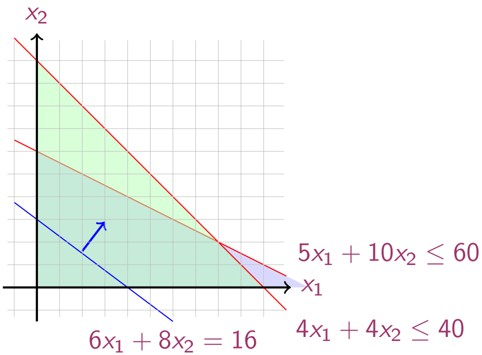
$$x_j \leq F_{max,j}, \qquad \forall j \in F$$

Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{aligned} \max\ 6x_1 + 8x_2 & \\ 5x_1 + 10x_2 &\leq 60 \\ 4x_1 + 4x_2 &\leq 40 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Graphical Representation:



In Matrix Form

$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = z \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \leq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{aligned}
\max \quad & z = \mathbf{c}^T\mathbf{x} \\
& A\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Linear Programming

Abstract mathematical model:
Parameters, Decision Variables, Objective, Constraints
(+ Domains & Quantifiers)

The Syntax of a Linear Programming Problem

objective func.	$\max / \min \mathbf{c}^T\mathbf{x}$	$\mathbf{c} \in \mathbb{R}^n$
constraints	s.t. $A\mathbf{x} \gtreqless \mathbf{b}$	$A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$
	$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \in \mathbb{R}^n, \mathbf{0} \in \mathbb{R}^n$

Essential features: continuity, linearity (proportionality and additivity), certainty of parameters

- Any vector $\mathbf{x} \in \mathbb{R}^n$ satisfying all constraints is a feasible solution.
- Each $\mathbf{x}^* \in \mathbb{R}^n$ that gives the best possible value for $\mathbf{c}^T\mathbf{x}$ among all feasible $\mathbf{x}$ is an optimal solution or optimum
- The value $\mathbf{c}^T\mathbf{x}^*$ is the optimum value

- The linear programming model consisted of 9 equations in 77 variables
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution
- The original instance: http://www.gams.com/modlib/libhtml/diet.htm

AMPL Model

```
# diet.mod
set NUTR;
set FOOD;

param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max { j in FOOD} >= f_min[j];
param n_min { NUTR } >= 0;
param n_max {i in NUTR } >= n_min[i];
param amt {NUTR,FOOD} >= 0;

var Buy { j in FOOD} >= f_min[j], <= f_max[j]

minimize total_cost: sum { j in FOOD } cost [j] * Buy[j];
subject to diet { i in NUTR }:
        n_min[i] <= sum {j in FOOD} amt[i,j] * Buy[j] <= n_max[i];
```

AMPL Model

```
# diet.dat
data;

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH MTL SPG TUR;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
   A 700 10000
   C 700 10000
   B1 700 10000
   B2 700 10000 ;
# %
```

```
param amt (tr):
           A C B1 B2 :=
   BEEF 60 20 10 15
   CHK 8 0 20 20
   FISH 8 10 15 10
   HAM 40 40 35 10
   MCH 15 35 15 15
   MTL 70 30 15 15
   SPG 25 50 25 15
   TUR 60 20 15 10 ;
```

Python Script

Model


```
# Model diet.py
m = Model("diet")

# Create decision variables for the foods to buy
buy = {}
for f in foods:
    buy[f] = m.addVar(obj=cost[f], name=f)

# The objective is to minimize the costs
m.modelSense = GRB.MINIMIZE

# Update model to integrate new variables
m.update()

# Nutrition constraints
for c in categories:
    m.addConstr(
      quicksum(nutritionValues[f,c] * buy[f] for f in foods) <= maxNutrition[c], name=c+'max')
    m.addConstr(
      quicksum(nutritionValues[f,c] * buy[f] for f in foods) >= minNutrition[c], name=c+'min')

# Solve
m.optimize()
```

Python Script
Data

```
from gurobipy import *

categories, minNutrition, maxNutrition = multidict({
  'calories': [1800, 2200],
  'protein': [91, GRB.INFINITY],
  'fat': [0, 65],
  'sodium': [0, 1779] })

foods, cost = multidict({
  'hamburger': 2.49,
  'chicken': 2.89,
  'hot dog': 1.50,
  'fries': 1.89,
  'macaroni': 2.09,
  'pizza': 1.99,
  'salad': 2.49,
  'milk': 0.89,
  'ice cream': 1.59 })
```

```
# Nutrition values for the foods
nutritionValues = {
  ('hamburger', 'calories'): 410,
  ('hamburger', 'protein'): 24,
  ('hamburger', 'fat'): 26,
  ('hamburger', 'sodium'): 730,
  ('chicken', 'calories'): 420,
  ('chicken', 'protein'): 32,
  ('chicken', 'fat'): 10,
  ('chicken', 'sodium'): 1190,
  ('hot dog', 'calories'): 560,
  ('hot dog', 'protein'): 20,
  ('hot dog', 'fat'): 32,
  ('hot dog', 'sodium'): 1800,
  ('fries', 'calories'): 380,
  ('fries', 'protein'): 4,
  ('fries', 'fat'): 19,
  ('fries', 'sodium'): 270,
  ('macaroni', 'calories'): 320,
  ('macaroni', 'protein'): 12,
  ('macaroni', 'fat'): 10,
  ('macaroni', 'sodium'): 930,
  ('pizza', 'calories'): 320,
  ('pizza', 'protein'): 15,
  ...
```

Outline

History of Linear Programming (LP)

System of linear equations

⇝ It is impossible to find out who knew what when first.
Just two "references":

- Egyptians and Babylonians considered about 2000 B.C. the solution of special linear equations. But, of course, they described examples and did not describe the methods in "today's style".

- What we call "Gaussian elimination"today has been explicitly described in Chinese "Nine Books of Arithmetic"which is a compendium written in the period 2010 B.C. to A.D. 9, but the methods were probably known long before that.

- Gauss, by the way, never described "Gaussian elimination". He just used it and stated that the linear equations he used can be solved "per eliminationem vulgarem"

History of Linear Programming (LP)

- Origins date back to Newton, Leibnitz, Lagrange, etc.
- In 1827, Fourier described a variable elimination method for systems of linear inequalities, today often called Fourier-Moutzkin elimination (Motzkin, 1937). It can be turned into an LP solver but inefficient.
- In 1932, Leontief (1905-1999) Input-Output model to represent interdependencies between branches of a national economy (1976 Nobel prize)
- In 1939, Kantorovich (1912-1986): Foundations of linear programming (Nobel prize in economics with Koopmans on LP, 1975) on Optimal use of scarce resources: foundation and economic interpretation of LP
- The math subfield of Linear Programming was created by George Dantzig, John von Neumann (Princeton), and Leonid Kantorovich in the 1940s.
- In 1947, Dantzig (1914-2005) invented the (primal) simplex algorithm working for the US Air Force at the Pentagon. (program=plan)

History of LP (cntd)

- In 1954, Lemke: dual simplex algorithm, In 1954, Dantzig and Orchard Hays: revised simplex algorithm
- In 1970, Victor Klee and George Minty created an example that showed that the classical simplex algorithm has exponential worst-case behavior.
- In 1979, L. Khachain found a new efficient algorithm for linear programming. It was terribly slow. (Ellipsoid method)
- In 1984, Karmarkar discovered yet another new efficient algorithm for linear programming. It proved to be a strong competitor for the simplex method. (Interior point method)

History of Optimization

- In 1951, Nonlinear Programming began with the Karush-Kuhn-Tucker Conditions
- In 1952, Commercial Applications and Software began
- In 1950s, Network Flow Theory began with the work of Ford and Fulkerson.
- In 1955, Stochastic Programming began
- In 1958, Integer Programming began by R. E. Gomory.
- In 1962, Complementary Pivot Theory

Outline

Fourier Motzkin elimination method

Has $Ax \leq b$ a solution? (Assumption: $A \in \mathbb{Q}^{m \times n}, \mathbf{b} \in \mathbb{Q}^n$)
Idea:

1. transform the system into another by eliminating some variables such that the two systems have the same solutions over the remaining variables.
2. reduce to a system of constant inequalities that can be easily decided

Let x_r be the variable to eliminate
Let $M = \{1 \ldots m\}$ index the constraints
For a variable j let's partition the rows of the matrix in

$$N = \{i \in M \mid a_{ij} < 0\}$$
$$Z = \{i \in M \mid a_{ij} = 0\}$$
$$P = \{i \in M \mid a_{ij} > 0\}$$

$$\begin{cases} x_r \geq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, & a_{ir} < 0 \\ x_r \leq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, & a_{ir} > 0 \\ \text{all other constraints} & i \in Z \end{cases} \qquad \begin{cases} x_r \geq A_i(x_1, \ldots, x_{r-1}), & i \in N \\ x_r \leq B_i(x_1, \ldots, x_{r-1}), & i \in P \\ \text{all other constraints} & i \in Z \end{cases}$$

Hence the original system is equivalent to

$$\begin{cases} \max\{A_i(x_1, \ldots, x_{r-1}), i \in N\} \leq x_r \leq \min\{B_i(x_1, \ldots, x_{r-1}), i \in P\} \\ \text{all other constraints} \quad i \in Z \end{cases}$$

which is equivalent to

$$\begin{cases} A_i(x_1, \ldots, x_{r-1}) \leq B_j(x_1, \ldots, x_{r-1}) & i \in N, j \in P \\ \text{all other constraints} & i \in Z \end{cases}$$

we eliminated x_r but:

$$\begin{cases} |N| \cdot |P| \text{ inequalities} \\ |Z| \text{ inequalities} \end{cases}$$

after d iterations if $|P| = |N| = m/2$ exponential growth: $(1/4^d)(m/2)^{2^d}$

Example

$$
\begin{aligned}
-7x_1 + 6x_2 &\leq 25 \\
x_1 - 5x_2 &\leq 1 \\
x_1 &\leq 7 \\
-x_1 + 2x_2 &\leq 12 \\
-x_1 - 3x_2 &\leq 1 \\
2x_1 - x_2 &\leq 10
\end{aligned}
$$

x_2 variable to eliminate
$N = \{2, 5, 6\}, Z = \{3\}, P = \{1, 4\}$
$|Z \cup (N \times P)| = 7$ constraints

By adding one variable and one inequality, Fourier-Motzkin elimination can be turned into an LP solver.

Outline

Definitions

- $\mathbb{R}$: set of real numbers
  $\mathbb{N} = \{1, 2, 3, 4, ...\}$: set of natural numbers (positive integers)
  $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$: set of all integers
  $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$: set of rational numbers

- column vector and matrices
  scalar product: $\mathbf{y}^T\mathbf{x} = \sum_{i=1}^{n} y_i x_i$

- linear combination

$$\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_k \in \mathbb{R}^n \qquad \boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_k]^T \in \mathbb{R}^k \qquad \mathbf{x} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i$$

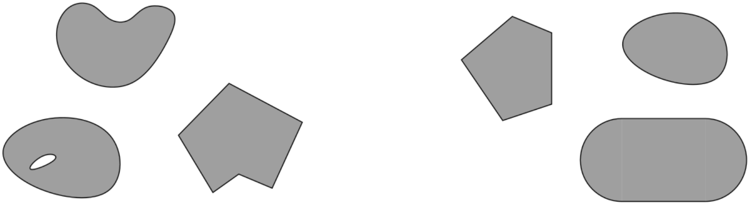
moreover:

$\boldsymbol{\lambda} \geq \mathbf{0}$	conic combination	
$\boldsymbol{\lambda}^T\mathbf{1} = 1$	affine combination	$\left(\sum_{i=1}^{k} \lambda_i = 1\right)$
$\boldsymbol{\lambda} \geq \mathbf{0}$ and $\boldsymbol{\lambda}^T\mathbf{1} = 1$	convex combination	

Definitions

- set S is linear (affine) independent if no element of it can be expressed as linear combination of the others
  Eg: $S \subseteq \mathbb{R}^n \implies$ max n lin. indep. (max $n+1$ aff. indep.)

- convex set: if $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq \lambda \leq 1$ then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$



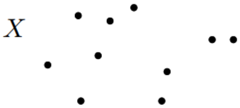
nonconvex convex

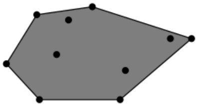
- convex function if its epigraph $\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is a convex set or $f : X \to \mathbb{R}$, or if $\forall x, y \in X, \lambda \in [0, 1]$ it holds that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

Definitions

- For a set of points $S \subseteq \mathbb{R}^n$

$\text{lin}(S)$	linear hull (span)
$\text{cone}(S)$	conic hull
$\text{aff}(S)$	affine hull
$\text{conv}(S)$	convex hull





the convex hull of X

$$\textbf{conv}(X) = \{\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_n x_n \mid \mathbf{x}_i \in X,\ \lambda_1, \ldots, \lambda_n \geq 0 \text{ and } \textstyle\sum_i \lambda_i = 1\}$$

Definitions

- rank of a matrix for columns (= for rows)
  if (m, n)-matrix has rank $= \min\{m, n\}$ then the matrix is full rank
  if (n, n)-matrix is full rank then it is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists\, \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

$$G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists\, \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} \leq \alpha\}$$

  ($\mathbf{a}^T\mathbf{x} = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subset \mathbb{R}^n$ is a polyhedron if $\exists\, m \in \mathbb{Z}^+, A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$:

$$P = \{\mathbf{x} \in \mathbb{R} \mid A\mathbf{x} \leq \mathbf{b}\} = \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_{i,\cdot}\mathbf{x} \leq b_i\}$$

  i.e., a polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

- a polyhedron P is a polytope if it is bounded: $\exists\, B \in \mathbb{R}, B > 0$:

$$P \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq B\}$$

- A set of vectors is a polytope if it is the convex hull of finitely many vectors.

Definitions

- General optimization problem: $\max\{\varphi(\mathbf{x}) \mid \mathbf{x} \in F\}$, F is feasible region for $\mathbf{x}$
- Note: if F is open, eg, $x < 5$ then: $\sup\{x \mid x < 5\}$
  sumpreum: least element of $\mathbb{R}$ greater or equal than any element in F
- If A and $\mathbf{b}$ are made of rational numbers, $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ is a rational polyhedron

Definitions

- The inequality denoted by $(\mathbf{a}, \alpha)$ is called a valid inequality for P if $\mathbf{ax} \leq \alpha, \forall x \in P$.
  Note that $(\mathbf{a}, \alpha)$ is a valid inequality if and only if P lies in the half-space $\{x \in \mathbb{R}^n \mid \mathbf{ax} \leq \alpha\}$.
- A face of P is $F = \{\mathbf{x} \in P \mid \mathbf{ax} = \alpha\}$ where $(\mathbf{a}, \alpha)$ is a valid inequality for P. Hence, it is the intersection of P with the hyperplane of a valid inequality. It is said to be proper if $F \neq \emptyset$ and $F \neq P$.
- If $F \neq \emptyset$ we say that it supports P.
  If c is a non zero vector for which $\delta = \max\{cx \mid x \in P\}$ is finite,
  then the set $\{x \mid cx = \delta\}$ is called supporting hyperplane.
- A point $\mathbf{x}$ for which $\{\mathbf{x}\}$ is a face is called a vertex of P and also a basic solution of $A\mathbf{x} \leq \mathbf{b}$ ($0\,\mathrm{dim}$ face)
- A facet is a maximal face distinct from P
  $\mathbf{cx} \leq \mathbf{d}$ is facet defining if $\mathbf{cx} = \mathbf{d}$ is a supporting hyperplane of P of $n-1\,\mathrm{dim}$

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$

Task:

1. decide that $\{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} \leq \mathbf{b}\}$ is empty (prob. infeasible), or
2. find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x}$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and $\exists$ solution
   1. one solution
   2. infinite solution
3. $F \neq \emptyset$ and $\nexists$ solution

Outline

Fundamental Theorem of LP

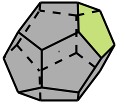
Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\} \;\; \textit{where } P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

If P is a bounded polyhedron and not empty and $\mathbf{x}^*$ is an optimal solution to the problem, then:

- $\mathbf{x}^*$ *is an extreme point (vertex) of P, or*
- $\mathbf{x}^*$ *lies on a face $F \subset P$ of optimal solutions*



Proof idea:

- assume $\mathbf{x}^*$ not a vertex of P then $\exists$ a ball around it still in P. Show that a point in the ball has better cost
- if $\mathbf{x}^*$ is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

Implications:

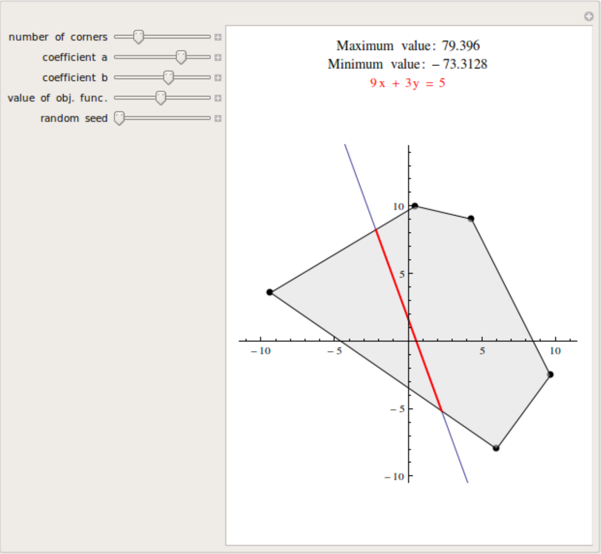
- the optimal solution is at the intersection of supporting hyperplanes.
- hence finitely many possibilities
- solution method: write all inequalities as equalities and solve all $\binom{m}{n}$ systems of linear equalities (n # variables, m # equality constraints)
- for each point we then need to check if feasible and if best in cost.
- each system is solved by Gaussian elimination
- Stirling approximation:

$$\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}} \text{ as } m \to \infty$$

Simplex Method

1. find a solution that is at the intersection of some n hyperplanes
2. try systematically to produce the other points by exchanging one hyperplane with another
3. check optimality, proof provided by duality theory

Demo


number of corners
coefficient a
coefficient b
value of obj. func.
random seed
Maximum value: 79.396
Minimum value: – 73.3128
9 x + 3 y = 5
10
5
– 5
– 10
– 10
– 5
5
10

Outline

Gaussian Elimination

1. Forward elimination
   reduces the system to row echelon form by elementary row operations
   - multiply a row by a non-zero constant
   - interchange two rows
   - add a multiple of one row to another

   (or LU decomposition)

2. Back substitution (or reduced row echelon form - RREF)

Example

$$
\begin{aligned}
2x + y - z &= 8 & (R1)\\
-3x - y + 2z &= -11 & (R2)\\
-2x + y + 2z &= -3 & (R3)
\end{aligned}
$$

```
|----+----+----+----+-----|
| R1 |  2 |  1 | -1 |   8 |
| R2 | -3 | -1 |  2 | -11 |
| R3 | -2 |  1 |  2 |  -3 |
|----+----+----+----+-----|
```

$$
\begin{aligned}
2x + y - z &= 8 & (R1)\\
+ \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 & (R2)\\
+ 2y + 1z &= 5 & (R3)
\end{aligned}
$$

```
|---------------+---+-----+------+---|
| R1'=1/2 R1    | 1 | 1/2 | -1/2 | 4 |
| R2'=R2+3/2 R1 | 0 | 1/2 | 1/2  | 1 |
| R3'=R3+R1     | 0 | 2   | 1    | 5 |
|---------------+---+-----+------+---|
```

$$
\begin{aligned}
2x + y - z &= 8 & (R1)\\
+ \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 & (R2)\\
- z &= 1 & (R3)
\end{aligned}
$$

```
|-------------+---+-----+------+---|
| R1'=R1      | 1 | 1/2 | -1/2 | 4 |
| R2'=2 R2    | 0 |   1 |    1 | 2 |
| R3'=R3-4 R2 | 0 |   0 |   -1 | 1 |
|-------------+---+-----+------+---|
```

$$
\begin{aligned}
2x + y - z &= 8 & (R1)\\
+ \tfrac{1}{2}y + \tfrac{1}{2}z &= 1 & (R2)\\
- z &= 1 & (R3)
\end{aligned}
$$

```
|---------------+---+-----+---+-----|
| R1'=R1-1/2 R3 | 1 | 1/2 | 0 | 7/2 |
| R2'=R2+R3     | 0 |   1 | 0 |   3 |
| R3'=-R3       | 0 |   0 | 1 |  -1 |
|---------------+---+-----+---+-----|
```

$$
\begin{aligned}
x &= 2 & (R1)\\
y &= 3 & (R2)\\
z &= -1 & (R3)
\end{aligned}
$$

```
|---------------+---+---+---+----+
| R1'=R1-1/2 R2 | 1 | 0 | 0 |  2 | => x=2
| R2'=R2        | 0 | 1 | 0 |  3 | => y=3
| R3'=R3        | 0 | 0 | 1 | -1 | => z=-1
|---------------+---+---+---+----+
```

LU Factorization

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = PLU$$

$$\mathbf{x} = A^{-1}\mathbf{b} = U^{-1}L^{-1}P^T\mathbf{b}$$

$$\mathbf{z}_1 = P^T\mathbf{b}, \quad \mathbf{z}_2 = L^{-1}\mathbf{z}_1, \quad \mathbf{x} = U^{-1}\mathbf{z}_2$$

```
In [36]: import pprint
    ...: import scipy
    ...:
    ...: A = scipy.array([ [2, 1, -1], [-3, -1, 2], [-2, 1, 2] ])
    ...: P,L,U = scipy.linalg.lu(A)
    ...:
    ...: print(P)
    ...: print(L)
    ...: print(U)
```

```
[[0. 0. 1.]
 [1. 0. 0.]
 [0. 1. 0.]]
[[ 1.          0.          0.        ]
 [ 0.66666667  1.          0.        ]
 [-0.66666667  0.2         1.        ]]
[[-3.         -1.          2.        ]
 [ 0.          1.66666667  0.66666667]
 [ 0.          0.          0.2       ]]
```

Polynomial time $O(n^2 m)$ but needs to guarantee that all the numbers during the run can be represented by polynomially bounded bits

Summary