DM545/DM554

Linear and Integer Programming

Lecture 5
Duality

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Outline

1. Derivation and Motivation

2. Theory

Outline

Dual Problem

Dual variables $\mathbf{y}$ in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad z &= \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= \mathbf{b}^T\mathbf{y} \\ A^T\mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

Bounding approach

$$
\begin{aligned}
z^* = \max\ 4x_1 &+ x_2 + 3x_3 \\
x_1 &+ 4x_2 \leq 1 \\
3x_1 &+ x_2 + x_3 \leq 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

a feasible solution is a lower bound but how good?
By tentatives:

$$
\begin{aligned}
(x_1, x_2, x_3) &= (1, 0, 0) \leadsto z^* \geq 4 \\
(x_1, x_2, x_3) &= (0, 0, 3) \leadsto z^* \geq 9
\end{aligned}
$$

What about upper bounds?

$$
\begin{array}{rcrl}
 & & 2 \cdot (\ x_1 + 4x_2\ ) & \leq 2 \cdot 1 \\
 & & + 3 \cdot (\ 3x_1 + x_2 + x_3\ ) & \leq 3 \cdot 3 \\
\hline
4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 11x_2 + 3x_3 & \leq 11 \\
 & & & \\
c^T x & \leq & y^T A x & \leq y^T b
\end{array}
$$

Hence $z^* \leq 11$. Is this the best upper bound we can find?

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$
\begin{array}{rl}
y_1 \cdot (\;\; x_1 + 4x_2 \qquad\qquad\quad) & \leq \; y_1(1) \\
y_2 \cdot (\; 3x_1 + \; x_2 + \qquad\quad x_3) & \leq \; y_2(3) \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 & \leq y_1 + 3y_2
\end{array}
$$

Coefficients

$$
\begin{aligned}
y_1 \;\; + 3y_2 &\geq 4 \\
4y_1 + \;\; y_2 &\geq 1 \\
y_2 &\geq 3
\end{aligned}
$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 \leq y_1 + 3y_2$ then to attain the best upper bound:

$$
\begin{aligned}
\min \;\; y_1 \;\; + 3y_2 & \\
y_1 \;\; + 3y_2 &\geq 4 \\
4y_1 + \;\; y_2 &\geq 1 \\
y_2 &\geq 3 \\
y_1, y_2 &\geq 0
\end{aligned}
$$

Multipliers Approach

$$
\begin{array}{c}
\pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1}
\end{array}
\left[
\begin{array}{cccc|cccc|c|c}
a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\
\vdots & \ddots & & & & & & & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 0 & b_m \\
\hdashline
c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0
\end{array}
\right]
$$

Working columnwise, since at optimum $\bar{c}_k \leq 0$ for all $k = 1, \dots, n+m$:

$$
\begin{cases}
\begin{array}{ccccccccc}
\pi_1 a_{11} & + & \pi_2 a_{21} & \dots + & \pi_m a_{m1} & + \pi_{m+1} c_1 & \leq & 0 \\
\vdots & \ddots & & & & & & \vdots \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots + & \pi_m a_{mn} & + \pi_{m+1} c_n & \leq & 0 \\
\hdashline
\pi_1 a_{1,n+1}, & & \pi_2 a_{2,n+1}, & \dots & \pi_m a_{m,n+1} & & \leq & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\pi_1 a_{1,n+m}, & & \pi_2 a_{2,n+m}, & \dots & \pi_m a_{m,n+m} & & \leq & 0 \\
\hdashline
& & & & & \pi_{m+1} & = & 1 \\
\hdashline
\pi_1 b_1 & + & \pi_2 b_2 & \dots + & \pi_m b_m & & & (\leq 0)
\end{array}
\end{cases}
$$

(since from the last row $z = -\boldsymbol{\pi}\mathbf{b}$ and we want to maximize z then we would $\min(-\boldsymbol{\pi}\mathbf{b})$ or equivalently $\max \boldsymbol{\pi}\mathbf{b}$)

$$\begin{array}{rl}
\max & \pi_1 b_1 + \pi_2 b_2 \ldots + \pi_m b_m \\
& \pi_1 a_{11} + \pi_2 a_{21} \ldots + \pi_m a_{m1} \leq -c_1 \\
& \vdots \quad \ddots \quad \vdots \\
& \pi_1 a_{1n} + \pi_2 a_{2n} \ldots + \pi_m a_{mn} \leq -c_n \\
& \pi_1, \pi_2, \ldots \pi_m \leq 0
\end{array}$$

$y = -\pi$

$$\begin{array}{rl}
\max & -y_1 b_1 + -y_2 b_2 \ldots + -y_m b_m \\
& -y_1 a_{11} + -y_2 a_{21} \ldots + -y_m a_{m1} \leq -c_1 \\
& \vdots \quad \ddots \quad \vdots \\
& -y_1 a_{1n} + -y_2 a_{2n} \ldots + -y_m a_{mn} \leq -c_n \\
& -y_1, -y_2, \ldots - y_m \leq 0
\end{array}$$

$$\begin{array}{rl}
\min & w = \mathbf{b}^T \mathbf{y} \\
& A^T \mathbf{y} \geq \mathbf{c} \\
& \mathbf{y} \geq \mathbf{0}
\end{array}$$

Example

$$\begin{aligned} \max\ 6x_1 &+ \ \ 8x_2 \\ 5x_1 &+ 10x_2 \le 60 \\ 4x_1 &+ \ \ 4x_2 \le 40 \\ &x_1, x_2 \ge \ 0 \end{aligned}$$

$$\begin{cases} 5\pi_1 + 4\pi_2 + 6\pi_3 \le 0 \\ 10\pi_1 + 4\pi_2 + 8\pi_3 \le 0 \\ 1\pi_1 + 0\pi_2 + 0\pi_3 \le 0 \\ 0\pi_1 + 1\pi_2 + 0\pi_3 \le 0 \\ 0\pi_1 + 0\pi_2 + 1\pi_3 = 1 \\ 60\pi_1 + 40\pi_2 \end{cases}$$

$$y_1 = -\pi_1 \ge 0$$
$$y_2 = -\pi_2 \ge 0$$

...

Duality Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

Outline

Symmetry

The dual of the dual is the primal:

Primal problem:

$$\begin{aligned} \max \quad z &= c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= b^T y \\ A^T y &\geq c \\ y &\geq 0 \end{aligned}$$

Let's put the dual in the standard form

Dual problem:

$$\begin{aligned} \min \quad b^T y &\equiv -\max -b^T y \\ -A^T y &\leq -c \\ y &\geq 0 \end{aligned}$$

Dual of Dual:

$$\begin{aligned} -\min \ -c^T x & \\ -Ax \quad &\geq \quad -b \\ x \quad &\geq \quad 0 \end{aligned}$$

Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

Theorem (Weak Duality Theorem)

Given:

$$(P)\ \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$
$$(D)\ \min\{\mathbf{b}^T\mathbf{y} \mid A^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$$

for any feasible solution $\mathbf{x}$ of (P) and any feasible solution $\mathbf{y}$ of (D):

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

Proof:
From (D) $c_j \leq \sum_{i=1}^{m} y_i a_{ij}\ \forall j$ and from (P) $\sum_{j=1}^{n} a_{ij}x_i \leq b_i\ \forall i$
From (D) $y_i \geq 0$ and from (P) $x_j \geq 0$

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_i\right) y_i \leq \sum_{i=1}^{m} b_i y_i$$

Strong Duality Theorem

Due to Von Neumann and Dantzig 1947 and Gale, Kuhn and Tucker 1951.

Theorem (Strong Duality Theorem)

Given:

$$(P)\ \max\{c^T x \mid Ax \leq b, x \geq 0\}$$
$$(D)\ \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

exactly one of the following occurs:

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution* $\mathbf{x}^* = [x_1^*, \ldots, x_n^*]$
   (D) has feasible solution $\mathbf{y}^* = [y_1^*, \ldots, y_m^*]$

$$\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.
- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)
- The last row of the final tableau will give us

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k = z^* + \sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i=1}^{m} \bar{c}_{n+i} x_{n+i} \qquad (*)$$

$$= z^* + \bar{c}_B x_B + \bar{c}_N x_N$$

In addition, $z^* = \sum_{j=1}^{n} c_j x_j^*$ because optimal value

- We define $\boxed{y_i^* = -\bar{c}_{n+i},}$ $i = 1, 2, \ldots, m$
- We claim that $(y_1^*, y_2^*, \ldots, y_m^*)$ is a dual feasible solution satisfying $c^T x^* = b^T y^*$.

- Let's verify the claim:
  We substitute in (*): i) $z = \sum_{j=1}^{n} c_j x_j$; ii) $\bar{c}_{n+i} = -y_i^*$; and iii) $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, 2, \ldots, m$ ($n+i$ are the slack variables)

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$
$$= \left(z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

This must hold for every $(x_1, x_2, \ldots, x_n)$ hence:

$$z^* = \sum_{i=1}^{m} b_i y_i^* \qquad \Longrightarrow \; y^* \text{ satisfies } c^T x^* = b^T y^*$$

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, j = 1, 2, \ldots, n$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \ldots, n+m$:

$$\bar{c}_j \leq 0 \rightsquigarrow \quad c_j - \sum_{i=1}^{m} y_i^* a_{ij} \leq 0 \rightsquigarrow \quad \sum_{i=1}^{m} y_i^* a_{ij} \geq c_j \qquad j = 1, 2, \ldots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow \quad y_i^* = -\bar{c}_{n+i} \geq 0, \qquad i = 1, 2, \ldots, m$$

$\implies y^*$ is also dual feasible solution

Complementary Slackness Theorem

Theorem (Complementary Slackness)

A feasible solution x^* for (P)
A feasible solution y^* for (D)
Necessary and sufficient conditions for optimality of both:

$$\left(c_j - \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j^* = 0, \quad j = 1, \ldots, n$$

If $x_j^* \neq 0$ then $\sum y_i^* a_{ij} = c_j$ (no surplus)
If $\sum y_i^* a_{ij} > c_j$ then $x_j^* = 0$

Proof:

$$z^* = \mathbf{c}^T \mathbf{x}^* \leq \mathbf{y}^* A \mathbf{x}^* \leq \mathbf{b}^T \mathbf{y}^* = w^*$$

Hence from strong duality theorem:

$$\mathbf{c}\mathbf{x}^* - \mathbf{y}^* A \mathbf{x}^* = 0$$

In scalars

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be $= 0$

Proof in scalar form:

$$c_j x_j^* \leq \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* \quad j = 1, 2, \ldots, n \quad \text{from feasibility in D}$$

$$\left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \leq b_i y_i^* \quad i = 1, 2, \ldots, m \quad \text{from feasibility in P}$$

Summing in j and in i:

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*$$

For the strong duality theorem the left hand side is equal to the right hand side and hence all inequalities become equalities.

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Duality - Summary

- Derivation:
  - Economic interpretation
  - Bounding Approach
  - Multiplier Approach
  - Recipe
  - Lagrangian Multipliers Approach (next time)
- Theory:
  - Symmetry
  - Weak Duality Theorem
  - Strong Duality Theorem
  - Complementary Slackness Theorem