DM545

Linear and Integer Programming

Lecture 6
More on Duality

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Outline

1. Derivation
    - Geometric Interpretation
    - Lagrangian Duality
    - Dual Simplex

2. Sensitivity Analysis

Summary

- Derivation:
  1. economic interpretation
  2. bounding
  3. multipliers
  4. recipe
  5. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
- Dual Simplex
- Sensitivity Analysis, Economic interpretation

Outline

Dual Problem

Dual variables $\mathbf{y}$ in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad z &= \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq 0 \end{aligned}$$

Dual Problem:

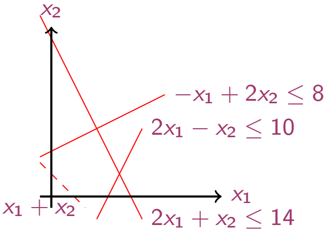
$$\begin{aligned} \min \quad w &= \mathbf{b}^T\mathbf{y} \\ A^T\mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\in \mathbb{R}^m \end{aligned}$$

- Basic feasible solutions of (P) give immediate lower bounds on the optimal value z^*. Is there a simple way to get upper bounds?
- The optimal solution must satisfy any linear combination $\mathbf{y} \in \mathbb{R}^m$ of the equality constraints.
- If we can construct a linear combination of the equality constraints $\mathbf{y}^T(A\mathbf{x}) = \mathbf{y}^T\mathbf{b}$, for $\mathbf{y} \in \mathbb{R}^m$, such that $\mathbf{c}^T\mathbf{x} \leq \mathbf{y}^T(A\mathbf{x})$, then $\mathbf{y}^T(A\mathbf{x}) = \mathbf{y}^T\mathbf{b}$ is an upper bound on z^*.

Outline

Geometric Interpretation

$$\begin{array}{rrrrl} \max & x_1 & + & x_2 & = z \\ & 2x_1 & + & x_2 & \leq 14 \\ & -x_1 & + & 2x_2 & \leq 8 \\ & 2x_1 & - & x_2 & \leq 10 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

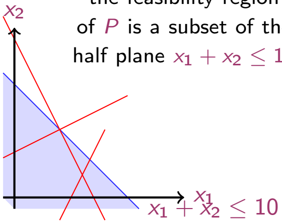


Feasible sol $x^* = (4, 6)$ yields $z^* = 10$. To prove that it is optimal we need to verify that $y^* = (3/5, 1/5, 0)$ is a feasible solution of D:

$$\begin{array}{rrrrl} \min & 14y_1 & + 8y_2 & + 10y_3 & = w \\ & 2y_1 & - y_2 & + 2y_3 & \geq 1 \\ & y_1 & + 2y_2 & - y_3 & \geq 1 \\ & & & y_1, y_2, y_3 & \geq 0 \end{array}$$

and that $w^* = 10$

$$\begin{array}{l} \frac{3}{5} \cdot (\ \ 2x_1 + \ \ x_2 \leq 14) \\ \frac{1}{5} \cdot (\ -x_1 + 2x_2 \leq \ \ 8) \\ \hline \qquad\quad x_1 + \ \ x_2 \leq \ 10 \end{array}$$



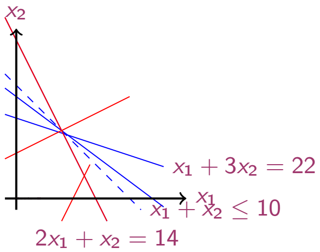
$$(2v - w)x_1 + (v + 2w)x_2 \le 14v + 8w$$

set of halfplanes that contain the feasibility region of P and pass through $[4, 6]$

$$2v - w \ge 1$$
$$v + 2w \ge 1$$

Example of boundary lines among those allowed:

$$v = 1, w = 0 \implies 2x_1 + x_2 = 14$$
$$v = 1, w = 1 \implies x_1 + 3x_2 = 22$$
$$v = 2, w = 1 \implies 3x_1 + 4x_2 = 36$$



Outline

Lagrangian Duality

Relaxation: if a problem is hard to solve then find an easier problem resembling the original one that provides information in terms of bounds. Then, search for the strongest bounds.

$$\begin{aligned} \min\ 13x_1 + 6x_2 + 4x_3 + 12x_4 & \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 7 \\ 3x_1 + + 2x_3 + 4x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

We wish to reduce to a problem easier to solve, ie:

$$\begin{aligned} \min\ c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ x_1, x_2, \ldots, x_n &\geq 0 \end{aligned}$$

solvable by inspection: if $c < 0$ then $x = +\infty$, if $c \geq 0$ then $x = 0$.
measure of violation of the constraints:

$$\begin{aligned} &7 - (2x_1 + 3x_2 + 4x_3 + 5x_4) \\ &2 - (3x_1 + + 2x_3 + 4x_4) \end{aligned}$$

We relax these measures in obj. function with Lagrangian multipliers y_1, y_2.
We obtain a family of problems:

$$PR(y_1, y_2) = \min_{x_1,x_2,x_3,x_4 \geq 0} \left\{ \begin{array}{rrrrr} & 13x_1 & + 6x_2 & + 4x_3 & + 12x_4 \\ +y_1(7- & 2x_1 & - 3x_2 & - 4x_3 & - 5x_4) \\ +y_2(2- & 3x_1 & & - 2x_3 & - 4x_4) \end{array} \right\}$$

1. for all $y_1, y_2 \in \mathbb{R} : \mathrm{opt}(PR(y_1, y_2)) \leq \mathrm{opt}(P)$
2. $\max_{y_1,y_2 \in \mathbb{R}} \{\mathrm{opt}(PR(y_1, y_2))\} \leq \mathrm{opt}(P)$

PR is easy to solve.
(It can be also seen as a proof of the weak duality theorem)

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{rl} & (13 - 2y_2 - 3y_2)\, x_1 \\ + & (6 - 3y_1 \qquad\quad)\, x_2 \\ + & (4 \qquad\quad - 2y_2)\, x_3 \\ + & (12 - 5y_1 - 4y_2)\, x_4 \\ + & \qquad 7y_1 + 2y_2 \end{array} \right\}$$

if coeff. of x is < 0 then bound is $-\infty$ then LB is useless

$$\begin{aligned} (13 - 2y_2 - 3y_2) &\geq 0 \\ (6 - 3y_1 \qquad\quad) &\geq 0 \\ (4 \qquad\quad - 2y_2) &\geq 0 \\ (12 - 5y_1 - 4y_2) &\geq 0 \end{aligned}$$

If they all hold then we are left with $7y_1 + 2y_2$ because all go to 0.

$$\begin{aligned} \max\ 7y_1 + 2y_2 & \\ 2y_2 + 3y_2 &\leq 13 \\ 3y_1 \qquad\quad &\leq 6 \\ + 2y_2 &\leq 4 \\ 5y_1 + 4y_2 &\leq 12 \end{aligned}$$

General Formulation

$$
\begin{aligned}
\min \quad & z = \mathbf{c}^T\mathbf{x} & \qquad & \mathbf{c} \in \mathbb{R}^n \\
& A\mathbf{x} = \mathbf{b} & & A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m \\
& \mathbf{x} \geq \mathbf{0} & & \mathbf{x} \in \mathbb{R}^n
\end{aligned}
$$

$$\max_{\mathbf{y}\in\mathbb{R}^m}\{\min_{\mathbf{x}\in\mathbb{R}^n_+}\{\mathbf{c}^T\mathbf{x} + \mathbf{y}^T(\mathbf{b} - A\mathbf{x})\}\}$$

$$\max_{\mathbf{y}\in\mathbb{R}^m}\{\min_{\mathbf{x}\in\mathbb{R}^n_+}\{(\mathbf{c}^T - \mathbf{y}^T A)\mathbf{x} + \mathbf{y}^T\mathbf{b}\}\}$$

$$
\begin{aligned}
\max \quad & \mathbf{b}^T\mathbf{y} \\
& A^T\mathbf{y} \leq \mathbf{c} \\
& \mathbf{y} \in \mathbb{R}^m
\end{aligned}
$$

Outline

Dual Simplex

- Dual simplex (Lemke, 1954): apply the simplex method to the dual problem and observe what happens in the primal tableau:

$$
\begin{aligned}
\max\{c^T x \mid Ax \leq b, x \geq 0\} &= \min\{b^T y \mid A^T y \geq c^T, y \geq 0\} \\
&= -\max\{-b^T y \mid -A^T x \leq -c^T, y \geq 0\}
\end{aligned}
$$

- We obtain a new algorithm for the primal problem: the dual simplex
  It corresponds to the primal simplex applied to the dual

Primal simplex on primal problem:

1. pivot > 0
2. col c_j with wrong sign
3. row: $\min\left\{\frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, .., m\right\}$

Dual simplex on primal problem:

1. pivot < 0
2. row $b_i < 0$
   (condition of feasibility)
3. col: $\min\left\{\left|\frac{c_j}{a_{ij}}\right| : a_{ij} < 0, j = 1, 2, .., n + m\right\}$
   (least worsening solution)

Dual Simplex

0. (primal) simplex on primal problem (the one studied so far)

1. Now: dual simplex on primal problem $\equiv$ primal simplex on dual problem
   (implemented as dual simplex, understood as primal simplex on dual problem)

Uses of 1.:

- The dual simplex can work better than the primal in some cases.
  Eg. since running time in practice between $2m$ and $3m$, then if $m = 99$ and $n = 9$ then better the dual

- Infeasible start
  Dual based Phase I algorithm (Dual-primal algorithm)

Dual Simplex for Phase I

Example

Primal:

$$\begin{aligned} \max \quad & -x_1 - x_2 \\ & -2x_1 - x_2 \leq 4 \\ & -2x_1 + 4x_2 \leq -8 \\ & -x_1 + 3x_2 \leq -7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Initial tableau

```
|   | x1 | x2 | w1 | w2 | w3 | -z |  b |
|---+----+----+----+----+----+----+----|
|   | -2 | -1 |  1 |  0 |  0 |  0 |  4 |
|   | -2 |  4 |  0 |  1 |  0 |  0 | -8 |
|   | -1 |  3 |  0 |  0 |  1 |  0 | -7 |
|---+----+----+----+----+----+----+----|
|   | -1 | -1 |  0 |  0 |  0 |  1 |  0 |
```

infeasible start

- x_1 enters, w_2 leaves

Dual:

$$\begin{aligned} \min \quad & 4y_1 - 8y_2 - 7y_3 \\ & -2y_1 - 2y_2 - y_3 \geq -1 \\ & -y_1 + 4y_2 + 3y_3 \geq -1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

- Initial tableau ($\min by \equiv -\max -by$)

```
|   | y1 | y2 | y3 | z1 | z2 | -p | b |
|---+----+----+----+----+----+----+---|
|   |  2 |  2 |  1 |  1 |  0 |  0 | 1 |
|   |  1 | -4 | -3 |  0 |  1 |  0 | 1 |
|---+----+----+----+----+----+----+---|
|   | -4 |  8 |  7 |  0 |  0 |  1 | 0 |
```

feasible start (thanks to $-x_1 - x_2$)

- y_2 enters, z_1 leaves

- x_1 enters, w_2 leaves

	x1	x2	w1	w2	w3	-z	b
	0	-5	1	-1	0	0	12
	1	-2	0	-0.5	0	0	4
	0	1	0	-0.5	1	0	-3
	0	-3	0	-0.5	0	1	4

- w_2 enters, w_3 leaves (note that we kept $c_j < 0$, ie, optimality)

	x1	x2	w1	w2	w3	-z	b
	0	-7	1	0	-2	0	18
	1	-3	0	0	-1	0	7
	0	-2	0	1	-2	0	6
	0	-4	0	0	-1	1	7

- y_2 enters, z_1 leaves

	y1	y2	y3	z1	z2	-p	b
	1	1	0.5	0.5	0	0	0.5
	5	0	-1	2	1	0	3
	-4	0	3	-12	0	1	-4

- y_3 enters, y_2 leaves

	y1	y2	y3	z1	z2	-p	b
	2	2	1	1	0	0	1
	7	2	0	3	1	0	3
	-18	-6	0	-7	0	1	-7

Summary

- Derivation:
  1. bounding
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  3. recipe
  4. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
- Dual Simplex
- Sensitivity Analysis, Economic interpretation

Outline

Economic Interpretation

$$
\begin{aligned}
\max\ & 5x_0 + 6x_1 + 8x_2 \\
& 6x_0 + 5x_1 + 10x_2 \leq 60 \\
& 8x_0 + 4x_1 + 4x_2 \leq 40 \\
& 4x_0 + 5x_1 + 6x_2 \leq 50 \\
& x_0, x_1, x_2 \geq 0
\end{aligned}
$$

final tableau:

$x0$	$x1$	$x2$	$s1$	$s2$	$s3$	$-z$	b
	0	1		0			5/2
	1	0		0			7
	0	0		1			2
$-1/5$	0	0	$-1/5$	0	-1		-62

- Which are the values of variables, the reduced costs, the shadow prices (or marginal prices), the values of dual variables?
- If one slack variable > 0 then overcapacity: $s_2 = 2$ then the second constraint is not tight
- How many products can be produced at most? at most m
- How much more expensive a product not selected should be?
  look at reduced costs: $c_j + \boldsymbol{\pi} \mathbf{a}_j > 0$
- What is the value of extra capacity of manpower? In +1 out +1/5

Economic Interpretation

Game: Suppose two economic operators:

- P owns the factory and produces goods
- D is in the market buying and selling raw material and resources
- D asks P to close and sell him all resources
- P considers if the offer is convenient
- D wants to spend least possible
- y are prices that D offers for the resources
- $\sum y_i b_i$ is the amount D has to pay to have all resources of P
- $\sum y_i a_{ij} \geq c_j$ total value to make j > price per unit of product
- P either sells all resources $\sum y_i a_{ij}$ or produces product j (c_j)
- without $\geq$ there would not be negotiation because P would be better off producing and selling

- ▶ at optimality the situation is indifferent (strong th.)
- ▶ resource 2 that was not totally utilized in the primal has been given value 0 in the dual. (complementary slackness th.) Plausible, since we do not use all the resource, likely to place not so much value on it.
- ▶ for product 0 $\sum y_i a_{ij} > c_j$ hence not profitable producing it. (complementary slackness th.)

Sensitivity Analysis
aka Postoptimality Analysis

Instead of solving each modified problem from scratch, exploit results obtained from solving the original problem.

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \le \mathbf{x} \le \mathbf{u}\} \qquad (*)$$

(I) changes to coefficients of objective function: $\max\{\tilde{\mathbf{c}}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}$ (primal)
$\mathbf{x}^*$ of (*) remains feasible hence we can restart the simplex from $\mathbf{x}^*$

(II) changes to RHS terms: $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \tilde{\mathbf{b}}, \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}$ (dual)
$\mathbf{x}^*$ optimal feasible solution of (*)
basic sol $\bar{\mathbf{x}}$ of (II): $\bar{\mathbf{x}}_N = \mathbf{x}^*_N$, $A_B\bar{\mathbf{x}}_B = \tilde{\mathbf{b}} - A_N\bar{\mathbf{x}}_N$
$\bar{\mathbf{x}}$ is dual feasible and we can start the dual simplex from there. If $\tilde{\mathbf{b}}$ differs from $\mathbf{b}$ only slightly it may be we are already optimal.

(III) introduce a new variable: (primal)

$$\max \quad \sum_{j=1}^{6} c_j x_j$$
$$\sum_{j=1}^{6} a_{ij} x_j = b_i, \;\; i = 1, \ldots, 3$$
$$l_j \le x_j \le u_j, \;\; j = 1, \ldots, 6$$
$$[x_1^*, \ldots, x_6^*] \text{ feasible}$$

$$\max \quad \sum_{j=1}^{7} c_j x_j$$
$$\sum_{j=1}^{7} a_{ij} x_j = b_i, \;\; i = 1, \ldots, 3$$
$$l_j \le x_j \le u_j, \;\; j = 1, \ldots, 7$$
$$[x_1^*, \ldots, x_6^*, 0] \text{ feasible}$$

(IV) introduce a new constraint: (dual)

$$\sum_{j=1}^{6} a_{4j} x_j = b_4$$
$$\sum_{j=1}^{6} a_{5j} x_j = b_5$$
$$l_j \le x_j \le u_j \qquad\qquad j = 7, 8$$

$$[x_1^*, \ldots, x_6^*] \text{ optimal}$$
$$[x_1^*, \ldots, x_6^*, x_7^*, x_8^*] \text{ feasible}$$
$$x_7^* = b_4 - \sum_{j=1}^{6} a_{4j} x_j^*$$
$$x_8^* = b_5 - \sum_{j=1}^{6} a_{5j} x_j^*$$

Examples

(I) Variation of reduced costs:

$$\begin{aligned} \max\ 6x_1 &+ \ 8x_2 \\ 5x_1 &+ 10x_2 \leq 60 \\ 4x_1 &+ \ 4x_2 \leq 40 \\ & x_1, x_2 \geq \ 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

The last tableau gives the possibility to estimate the effect of variations

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	2
x_1	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

For a variable in basis the perturbation goes unchanged in the red. costs. Eg:

$$\max\ (6+\delta)x_1 + 8x_2 \implies \bar{c}_1 = 1(6+\delta) - \frac{2}{5}\cdot 5 - 1\cdot 4 = \delta$$

then need to bring in canonical form and hence δ changes the obj value.
For a variable not in basis, if it changes the sign of the reduced cost $\implies$ worth bringing in basis
$\implies$the δ term propagates to other columns

(II) Changes in RHS terms

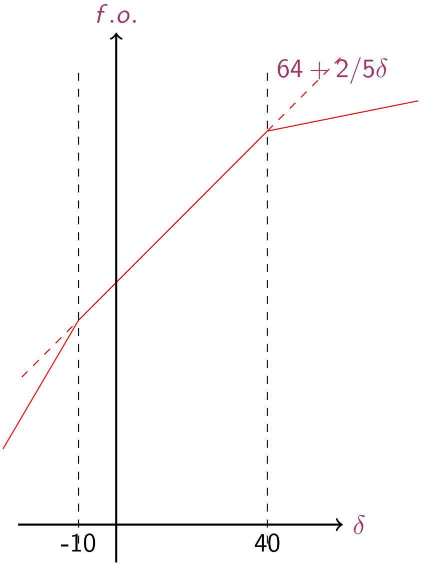
	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	$60+\delta$
x_4	4	4	0	1	0	$40+\epsilon$
	6	8	0	0	1	0

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	$2+1/5\delta-1/4\epsilon$
x_1	1	0	$-1/5$	$1/2$	0	$8-1/5\delta+1/2\epsilon$
	0	0	$-2/5$	-1	1	$-64-2/5\delta-\epsilon$

(It would be more convenient to augment the second. But let's take $\epsilon = 0$.)
If $60+\delta$ $\Longrightarrow$all RHS terms change and we must check feasibility
Which are the multipliers for the first row? $k_1 = \frac{1}{5}$, $k_2 = -\frac{1}{4}$, $k_3 = 0$

I: $1/5(60+\delta) - 1/4 \cdot 40 + 0 \cdot 0 = 12 + \delta/5 - 10 = 2 + \delta/5$
II: $-1/5(60+\delta) + 1/2 \cdot 40 + 0 \cdot 0 = -60/5 + 20 - \delta/5 = 8 - 1/5\delta$
Risk that RHS becomes negative
Eg: if $\delta = -10$ $\Longrightarrow$tableau stays optimal but not feasible $\Longrightarrow$apply dual simplex

Graphical Representation



(III) Add a variable

$$\begin{aligned} \max\ 5x_0 + 6x_1 + \ 8x_2 & \\ 6x_0 + 5x_1 + 10x_2 &\leq 60 \\ 8x_0 + 4x_1 + \ 4x_2 &\leq 40 \\ x_0, x_1, x_2 &\geq \ 0 \end{aligned}$$

Reduced cost of x_0? $c_j + \sum \pi_i a_{ij} = +1 \cdot 5 - \frac{2}{5} \cdot 6 + (-1)8 = -\frac{27}{5}$

To make worth entering in basis:

- increase its cost
- decrease the technological coefficient in constraint II: $5 - 2/5 \cdot 6 - a_{20} > 0$

(IV) Add a constraint

$$
\begin{array}{rl}
\max 6x_1 + 8x_2 & \\
5x_1 + 10x_2 & \leq 60 \\
4x_1 + 4x_2 & \leq 40 \\
5x_1 + 6x_2 & \leq 50 \\
x_1, x_2 & \geq 0
\end{array}
$$

Final tableau not in canonical form, need to iterate with dual simplex

	x_1	x_2	x_3	x_4	x_5	$-z$	b
x_2	0	1	$1/5$	$-1/4$		0	2
x_1	1	0	$-1/5$	$1/2$		0	8
	0	0	$5/5$	$6/4$	1	0	-2
	0	0	$-2/5$	-1	0	1	-64

(V) change in a technological coefficient:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	$10+\delta$	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

- first effect on its column
- then look at c
- finally look at b

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	$(10+\delta)1/5+4(-1/4)$	$1/5$	$-1/4$	0	2
x_1	1	$(10+\delta)(-1/5)+4(1/2)$	$-1/5$	$1/2$	0	8
	0	$-2/5\delta$	$-2/5$	-1	1	-64

Relevance of Sensitivity Analysis

- The dominant application of LP is mixed integer linear programming.
- In this context it is extremely important being able to begin with a model instantiated in one form followed by a sequence of problem modifications
  - row and column additions and deletions,
  - variable fixings

  interspersed with resolves

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