DM545

Linear and Integer Programming

Lecture 8
More on Polyhedra and Farkas Lemma

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

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1. Farkas Lemma
2. Beyond the Simplex

We now look at Farkas Lemma with two objectives:

- (giving another proof of strong duality)
- understanding a certificate of infeasibility

Farkas Lemma

Lemma (Farkas)

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then,

either	*I.*	$\exists \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$
or	*II.*	$\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y}^T A \geq 0^T$ *and* $\mathbf{y}^T \mathbf{b} < \mathbf{0}$

Easy to see that both I and II cannot occur together:

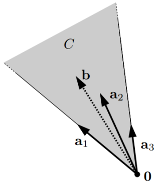
$$(0 \leq) \qquad \mathbf{y}^T A\mathbf{x} = \mathbf{y}^T \mathbf{b} \qquad (< 0)$$

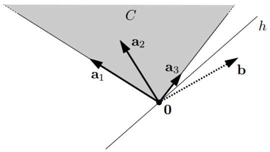
Geometric interpretation of Farkas L.

Linear combination of $\mathbf{a}_i$ with nonnegative terms generates a convex cone:

$$\{\lambda_1\mathbf{a}_1 + \ldots + \lambda_n\mathbf{a}_n, \mid \lambda_1, \ldots, \lambda_n \geq \mathbf{0}\}$$

Polyhedral cone: $C = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, intersection of many $\mathbf{ax} \leq 0$
Convex hull of rays $\mathbf{p}_i = \{\lambda_i\mathbf{a}_i, \lambda_i \geq 0\}$





Either	point $\mathbf{b}$ lies in convex cone C
or	$\exists$ hyperplane h passing through point 0 $h = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{y}^T\mathbf{x} = 0\}$ for $\mathbf{y} \in \mathbb{R}^m$ such that all vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ (and thus C) lie on one side and $\mathbf{b}$ lies (strictly) on the other side (ie, $\mathbf{y}^T\mathbf{a}_i \geq 0, \forall i = 1 \ldots n$ and $\mathbf{y}^T\mathbf{b} < 0$).

Variants of Farkas Lemma

Corollary

(i) $A\mathbf{x} = \mathbf{b}$ *has sol* $\mathbf{x} \geq \mathbf{0} \iff \forall \mathbf{y} \in \mathbb{R}^m$ *with* $\mathbf{y}^T A \geq \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq \mathbf{0}$

(ii) $A\mathbf{x} \leq \mathbf{b}$ *has sol* $\mathbf{x} \geq \mathbf{0} \iff \forall \mathbf{y} \geq \mathbf{0}$ *with* $\mathbf{y}^T A \geq \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq \mathbf{0}$

(iii) $A\mathbf{x} \leq \mathbf{0}$ *has sol* $\mathbf{x} \in \mathbb{R}^n \iff \forall \mathbf{y} \geq \mathbf{0}$ *with* $\mathbf{y}^T A = \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq \mathbf{0}$

i) $\implies$ ii):
$\bar{A} = [A \mid I_m]$
$A\mathbf{x} \leq \mathbf{b}$ has sol $\mathbf{x} \geq \mathbf{0} \iff \bar{A}\bar{\mathbf{x}} = \mathbf{b}$ has sol $\bar{\mathbf{x}} \geq \mathbf{0}$
By (i):

$\forall \mathbf{y} \in \mathbb{R}^m$
$\mathbf{y}^T\mathbf{b} \geq \mathbf{0}, \mathbf{y}^T\bar{A} \geq \mathbf{0}$

$\mathbf{y}^T A \geq \mathbf{0}$
$\mathbf{y} \geq \mathbf{0}$

relation with Fourier & Moutzkin method

	The system $A\mathbf{x} \leq \mathbf{b}$	The system $A\mathbf{x} = \mathbf{b}$
has a solution $\mathbf{x} \geq \mathbf{0}$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A \geq \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A \geq \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$
has a solution	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A = \mathbf{0}$	$\mathbf{y}^T A = \mathbf{0}^T$

Certificate of Infeasibility

Farkas Lemma provides a way to certificate infeasibility.

Theorem

Given a certificate $\mathbf{y}^*$ it is easy to check the conditions (by linear algebra):

$$\begin{aligned} A^T\mathbf{y}^* &\geq \mathbf{0} \\ \mathbf{b}\mathbf{y}^* &< 0 \end{aligned}$$

Why would $\mathbf{y}^*$ be a certificate of infeasibility?
Proof (by contradiction)
Assume, $A^T\mathbf{y}^* \geq \mathbf{0}$ and $\mathbf{b}\mathbf{y}^* < 0$.
Moreover assume $\exists \mathbf{x}^*$: $A\mathbf{x}^* = \mathbf{b}$, $\mathbf{x}^* \geq \mathbf{0}$,then:

$$(\geq 0) \qquad (\mathbf{y}^*)^T A\mathbf{x}^* = (\mathbf{y}^*)^T\mathbf{b} \qquad (< 0)$$

Contradiction

General form:

$$\begin{aligned} \max\ & c^T x \\ A_1 x &= b_1 \\ A_2 x &\leq b_2 \\ A_3 x &\geq b_3 \\ x &\geq 0 \end{aligned}$$

infeasible $\Leftrightarrow \exists y^*$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 + b_3^T y_3 &> 0 \\ A_1^T y_1 + A_2^T y_2 + A_3^T y_3 &\leq 0 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \end{aligned}$$

Example

$$\begin{aligned} \max\ & c^T x \\ x_1 &\leq 1 \\ x_1 &\geq 2 \end{aligned}$$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 &> 0 \\ A_1^T y_1 + A_2^T y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 &> 0 \\ y_1 + y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$y_1 = -1, y_2 = 1$ is a valid certificate.

- Observe that it is not unique!
- It can be reported in place of the dual solution because same dimension.
- To repair infeasibility we should change the primal at least so much as that the certificate of infeasibility is no longer valid.
- Only constraints with $y_i \neq 0$ in the certificate of infeasibility cause infeasibility

Duality: Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
  - Farkas Lemma:
    Strong duality + Infeasibility certificate
- Dual Simplex
- Economic interpretation
- Geometric Interpretation
- Sensitivity analysis

Resume

Advantages of considering the dual formulation:

- proving optimality (although the simplex tableau can already do that)
- gives a way to check the correctness of results easily
- alternative solution method (ie, primal simplex on dual)
- sensitivity analysis
- solving P or D we solve the other for free
- certificate of infeasibility

Outline

Interior Point Algorithms

- Ellipsoid method: cannot compete in practice but weakly polynomial time (Khachyian, 1979)
- Interior point algorithm(s) (Karmarkar, 1984) competitive with simplex and polynomial in some versions
  - affine scaling algorithm (Dikin)
  - logarithmic barrier algorithm (Fiacco and McCormick) $\equiv$ Karmakar's projective method

1. Start at an interior point of the feasible region
2. Move in a direction that improves the objective function value at the fastest possible rate while ensuring that the boundary is not reached
3. Transform the feasible region to place the current point at the center of it

- because of patents reasons, now mostly known as barrier algorithms
- one single iteration is computationally more intensive than the simplex (matrix calculations, sizes depend on number of variables)
- particularly competitive in presence of many constraints (eg, for $m = 10,000$ may need less than 100 iterations)
- bad for post-optimality analysis $\rightsquigarrow$ crossover algorithm to convert a solution of barrier method into a basic feasible solution for the simplex