DM559

Linear and Integer Programming

Lecture 1
Introduction

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Outline

Outline

Aims of the course

Learn about:

- both the theory and the practice of Linear Algebra
- one of the most important applications of Linear Algebra:
  - Mathematical optimization: linear programming
  - Discrete optimization: integer programming

$\rightsquigarrow$ Practical experience with computer software

$\rightsquigarrow$ The solution to really many real-life problems will be within short reach

Contents of the Course (1/2)

(see Syllabus)

Linear Algebra: manipulation of matrices and vectors with some theoretical background

Linear Algebra

1 Matrices and vectors - Matrix algebra, Geometric insight
2 Systems of Linear Equations - Gaussian elimination
3 Matrix inversion and determinants
4 Vector spaces
5 Linear Transformations - Matrix representation
6 Diagonalization - Eigenvalues and Eigenvectors
(7 Orthogonality or Numerical Methods)

Contents of the Course (2/2)

(see Syllabus)

Linear Programming

1	Introduction - Linear Programming, Notation
2	Linear Programming, Simplex Method
3	Exception Handling
4	Duality Theory
5	Sensitivity
6	Revised Simplex Method

Integer Linear Programming

7	Modeling Examples, Good Formulations, Relaxations
8	Well Solved Problems
9	Network Optimization Models (Max Flow, Min cost flow, matching)
10	Cutting Planes & Branch and Bound
11	More on Modeling

Practical Information

Teacher: Marco Chiarandini (http://www.imada.sdu.dk/~marco)
Instructor: Kristoffer Abell

Schedule:

- Introductory classes: 44 hours (22 classes)
- Training classes: 50 hours
  - Exercises: 42 hours
  - Laboratory: 8 hours

Alternative views of the schedule:

- http://mitsdu.sdu.dk/skema, SDU Mobile
- Official course description (læserplaner)
- http://www.imada.sdu.dk/~marco/DM559

Communication Means

- BlackBoard (BB) $\Leftrightarrow$ Main Web Page (WWW) (link http://www.imada.sdu.dk/~marco/DM559)
- Announcements in BlackBoard
- Ask peers
- You are welcome to visit me in my office in working hours (8-16)
- Write to Marco or to instructor
- midway evaluation

$\rightsquigarrow$ It is good to ask questions!!

$\rightsquigarrow$ Let me know when you think we should do things differently! Things can be changed.

Books

Linear Algebra Part:

AR Howard Anton and Chris Rorres. Elementary Linear Algebra. 11th Edition. 2014. Wiley

Other books:

AH Martin Anthony and Michele Harvey, Linear Algebra, Concepts and Methods. 2012. Cambridge

Le Steven J. Leon, Linear Algebra with Applications, 8th edition, Prentice Hall (2010).

FSV Computing with Python: An introduction to Python for science and engineering Claus Führer, Jan Erik Solem, Olivier Verdier

Books

Linear and Integer Programming Part:

LN Lecture Notes

MG J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

Wo L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

Other books:

HL Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

- ...

Course Material

Main Web Page (WWW) is the main reference for list of contents (ie[1], syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- software

[1] ie = id est, eg = exempli gratia, wrt = with respect to

Assessment

- 7.5 ECTS

- Obligatory Assignments, pass/fail, evaluation by teacher
  both theoretical and practical (programming) exercises

- 4 hour written exam, 7-grade scale, external censor
  (theory part)
  similar to exercises in class and past exams
  in June

- (language: Danish and/or English)

Obligatory Assignments

- Small practical tasks must be passed to attend the written exam
- Individual work
- They require the use of Python + a MILP Solver (2nd part)
  See Tools from Public Web Page

Training Sessions

- Prepare them in advance to get out the most
- Best carried out in small groups

Outline

Sets

- A set is a collection of objects. eg.: $A = \{1, 2, 3\}$
- $A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}$
  ('$|$' reads 'such that')
- $B = \{x \mid x \text{ is a student of this course}\}$
- $x \in A$
  x belongs to A
- set of no members: empty set, denoted $\emptyset$
- if a set S is a (proper) subset of a set T, we write ($S \subset T$) $T \subseteq S$
  $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 30\}$
- for two sets A and B, the union $A \cup B$ is $\{x \mid x \in A \text{ or } x \in B\}$
- for two sets A and B, the intersection $A \cap B$ is $\{x \mid x \in A \text{ and } x \in B\}$
  $\{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cap B = \{2, 5\}$

Numbers

- set of real numbers: $\mathbb{R}$
- set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, ...\}$ (positive integers); $\mathbb{N}_0$ to include zero
- set of all integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$; $\mathbb{Z}_0^+$ only positives and zero
- set of rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
- set of complex numbers: $\mathbb{C}$
- absolute value (non-negative):

  $$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

  $|a + b| \leq |a| + |b|$, $a, b \in \mathbb{R}$
- the set $\mathbb{R}^2$ is the set of ordered pairs (x, y) of real numbers
  (eg, coordinates of a point wrt a pair of axes, the Cartesian plane)

Matrices and Vectors

- A matrix is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- An $n \times 1$ matrix is a column vector, or simply a vector:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- the set $\mathbb{R}^n$ is the set of vectors $[x_1, x_2, \ldots, x_n]^T$ of real numbers (eg, coordinates of a point wrt an n-dimensional space, the Euclidean Space)

Basic Algebra

Elementary Algebra: the study of symbols and the rules for manipulating symbols. It differs from arithmetic in the use of abstractions, such as using letters to stand for numbers that are either unknown or allowed to take on many values

- collecting up terms: eg. $2a + 3b - a + 5b = a + 8b$
- multiplication of variables: eg:

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms: eg:

$$\begin{aligned} -(a - 2b) &= -a + 2b, \\ (2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2 \end{aligned}$$

- $a^r a^s = a^{r+s}, \quad (a^r)^s = a^{rs}, \quad a^{-n} = 1/a^n,$
  $a^{1/n} = x \iff x^n = a, \quad a^{m/n} = (a^{1/n})^m$

Variables

- In Mathematics and Statistics, a variable is an alphabetic character representing a value, which is unknown. They are used in symbolic calculations.
  Commonly given one-character names.

- in contrast, a constant or given or scalar is a known real number

- in contrast, Computer Science, a variable is a storage location paired with an associated identifier, which contains a value, which may be known or unknown.
  Commonly given long, explanatory names.

Functions

- a function f on a set $\mathcal{X}$ into a set $\mathcal{Y}$ is a rule that assigns a unique element $f(x)$ in S to each element x in $\mathcal{X}$.

$$y = f(x)$$

y dependent variable

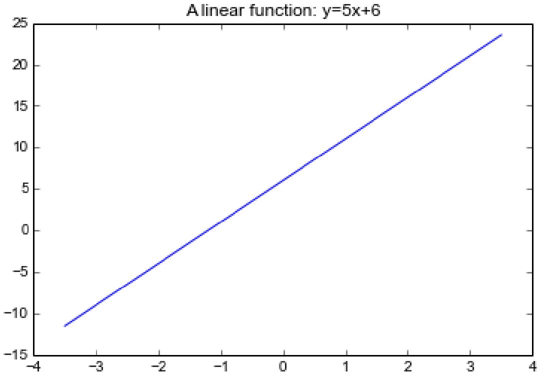
x independent variable

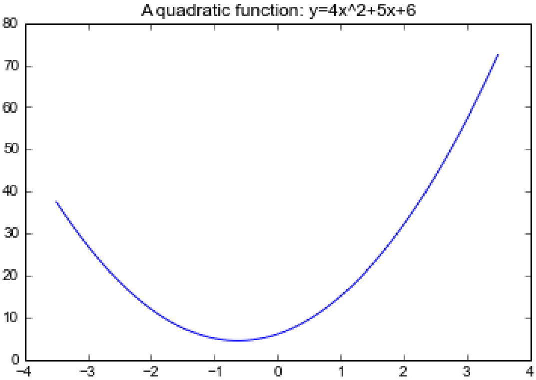
- a linear function has only sums and scalar multiplications, that is, for variable $x \in \mathbb{R}$ and scalars $a, b \in \mathbb{R}$:

$$f(x) := ax + b$$

Graphs of Functions

The graph of a function f consists of those points in the Cartesian plane whose coordinates (x, y) are pairs of input-output values for f.





Equations

- for a linear equation: $ax + b = 0$ with $a, b \in \mathbb{R}$
  the solution is a real number x for which the equation is true

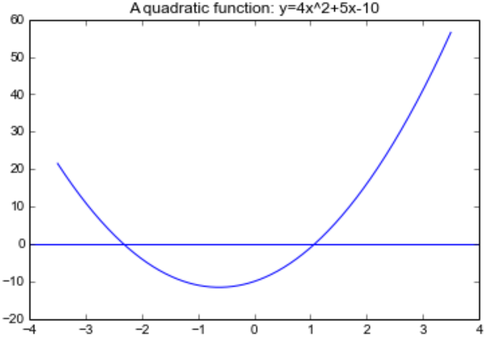
- Quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

- closed form or analytical solution:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$



- Quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

- quadratic formula:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

the term $b^2 - 4ac$ is called discriminant

- Solutions from discriminant:
  - if $b^2 - 4ac > 0 \implies$ two real solutions
  - if $b^2 - 4ac = 0 \implies$ exactly one solution: $x = -b/(2a)$
  - if $b^2 - 4ac < 0 \implies$ no real solution but complex solutions

- Can be solved also by factorization, eg:

$$x^2 - 6x + 5 = (x - 1)(x - 5) = 0$$

then either $x - 1 = 0$ or $x - 5 = 0$.
or by completing the square $\implies a(x + d)^2 + e = 0$, eg:
i) $x^2 + 6x + 9 = 0$, and ii) $x^2 - 2x + 3 = 0$

Polynomial Equations

- A polynomial of degree n in x is an expression of the form:

  $$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

  where the a_i are real constants, $a_n \neq 0$, and x is a real variable.

- $P_n(x) = 0$ has at most n solutions, eg:

  $$x^3 - 7x + 6 = (x-1)(x-2)(x+3) = 0,$$

  which are called roots or zeros of $P_n(x)$

- No general (closed) formula to find these roots

- If α is a solution then $(x - \alpha)$ must be a factor for $P_n(x)$
  We find α by trial and error and then set $(x-\alpha)Q(x)$ where $Q(x)$ is a polynomial of degree $n-1$

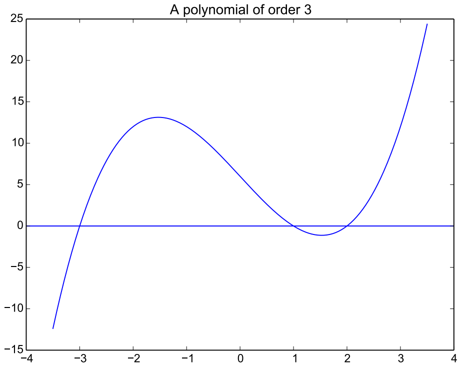
- Eg, $x^3 - 7x + 6$

In Python

```
import matplotlib.pyplot as plt
import numpy as np
a=[1,0,-7,6]
P=np.poly1d(a)
print(P)
```

```
   3
1 x - 7 x + 6
```

```
x = np.linspace(-3.5, 3.5, 500)
plt.plot(x, P(x), '-')
plt.axhline(y=0)
plt.title('A polynomial of order 3');
```



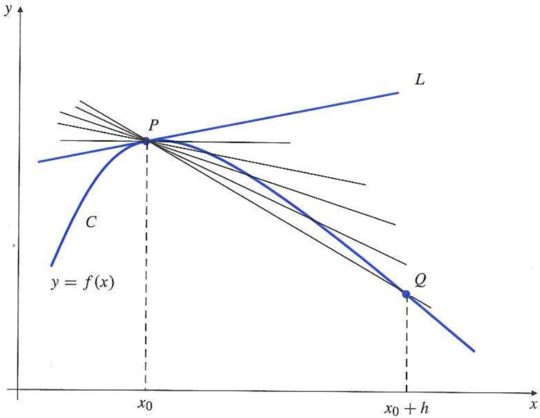
Differentiation

A line L through a point $(x_0, f(x_0))$ of f can be described by:

$$y = m(x - x_0) + f(x_0)$$

The derivative is the slope of the line that is tangent to the curve:

$$y = f'(x_0)(x - x_0) + f(x_0)$$



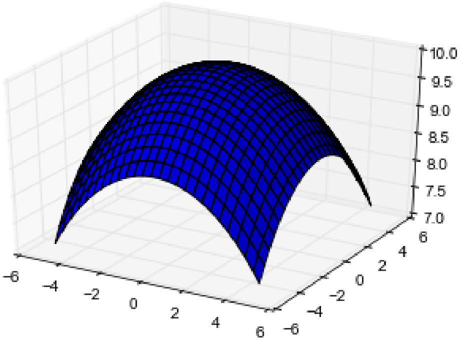
Functions of Several Variables

- A function f of n real variables is a rule that assigns a unique real number $f(x_1, x_2, \ldots, x_n)$ to each point $(x_1, x_2, \ldots, x_n)$

Example in $\mathbb{R}^2$:

$$f(x, y) = \sqrt{10^2 - x^2 - y^2}$$

$$x^2 + y^2 + z^2 = 10$$



Partial Derivatives

- The first partial derivative of the function $f(x, y)$ with respect to the variables x and y are:

$$f_1(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial}{\partial x} f(x, y)$$

$$f_2(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} = \frac{\partial}{\partial y} f(x, y)$$

- Their value in a point (x_0, y_0) is given by:

$$f_1(x_0, y_0) = \left(\frac{\partial}{\partial x} f(x, y) \right) \bigg|_{(x_0, y_0)}$$

$$f_2(x_0, y_0) = \left(\frac{\partial}{\partial y} f(x, y) \right) \bigg|_{(x_0, y_0)}$$

They identify the plane tangent to the function in the point (x_0, y_0).

Trigonometry

- sine and cosine functions, $\sin\theta$ and $\cos\theta$, recall the geometrical meaning
- angles measured in radiants rather than degrees ($\pi = 3.141...,\pi = 180$)
- $\cos x = \sin(x + \pi/2)$
- $\sin^2\theta + \cos^2\theta = 1$
- $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
- $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$