DM559

Linear and Integer Programming

# Lecture 2
# Systems of Linear Equations


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# Outline

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# A Motivating Example

You are organizing the next party at IMADA. In order to make it a memorable experience for everybody, you want to prepare 8 liters of a drink that contains 32% of alcohol.
At the store you find two products that contain 50% alcohol and 10% alcohol, respectively.
How much of each is needed?

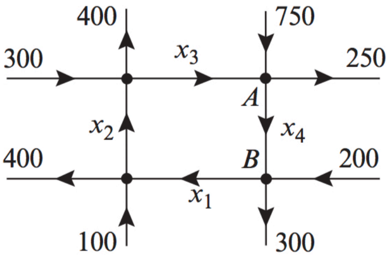
- Let $x$ represent the amount of the 50% product needed.
- Let $y$ represent the amount of the 10% product needed.
- The total amount of the mixture must be 8 liters. $x + y = 8$
- The amount of alcohol from each product in the end result must be 32% of 8 liters, or 0.32(8). $0.50x + 0.10y = 0.32(8)$
- We have a system of two linear equations and two variables (aka, unknowns).

$$\begin{cases} x + y = 8 \\ 0.50x + 0.10y = 0.32(8) \end{cases}$$

- How do you find a solution?
- Is there always a solution?

# Another Motivating Example

The picture shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour. Determine the flow rates in the inner branches of the network.




| Intersection | In Flow | | Out Flow |
|---|---|---|---|
| A: | $750 + x_3$ | $=$ | $250 + x_4$ |
| B: | $200 + x_4$ | $=$ | $300 + x_1$ |
| C: | $200 + x_1$ | $=$ | $400 + x_2$ |
| D: | $300 + x_2$ | $=$ | $400 + x_3$ |

$$\begin{cases} -x_3 + x_4 = 500 \\ -x_1 + x_4 = 100 \\ x_1 - x_2 = 200 \\ x_2 - x_3 = 100 \end{cases}$$

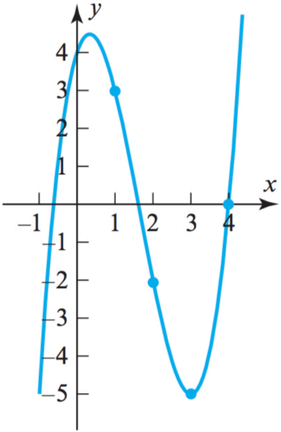
A system of linear equations in 4 unknowns

# Yet Another Motivating Example



**Polynomial Interpolation**

Given any $n$ points in the $xy$-plane that have distinct $x$-coordinates, there is a unique polynomial of degree $n-1$ or less whose graph passes through those points.

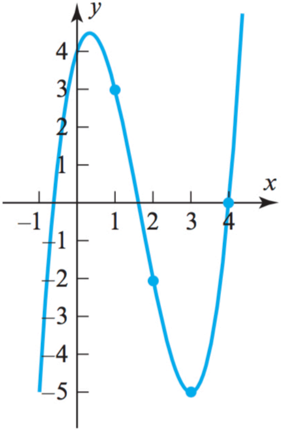



The graph of the polynomial is the graph of the equation:

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

The coordinates of the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ must satisfy:

$$\begin{cases} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} = y_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots = \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} = y_n \end{cases}$$




Find a cubic polynomial that passes through the points:

$$(1, 3) \quad (2, -2) \quad (3, -5) \quad (4, 0)$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 3 \\ a_0 + a_1 2 + a_2 4 + a_3 8 = -2 \\ a_0 + a_1 3 + a_2 9 + a_3 27 = -5 \\ a_0 + a_1 4 + a_2 16 + a_3 64 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \\ 0 \end{bmatrix}$$

# Systems of Linear Equations



## Definition (Linear Equation)

A linear equation in the $n$ variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad a_1, a_2, \ldots, a_n, b \in \mathbb{R}, \quad \exists i : a_i \neq 0$$

## Definition (System of linear equations, aka linear system)

A system of $m$ linear equations in $n$ unknowns $x_1, x_2, \ldots, x_n$ is a set of $m$ equations of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

The numbers $a_{ij}$ are known as the coefficients of the system.

We say that $s_1, s_2, \ldots, s_n$ is a solution of the system if all $m$ equations hold true when

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

# Examples



$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 4 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & + & x_5 & = & 5 \\ x_1 & & & & & + & x_4 & + & x_5 & = & 4 \end{array}$$

has solution

$$x_1 = -1, x_2 = -2, x_3 = 1, x_4 = 3, x_5 = 2.$$

Is it the only one?

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 4 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & + & x_5 & = & 5 \\ x_1 & & & & & + & x_4 & + & x_5 & = & 6 \end{array}$$

has no solutions

# Matrices



## Definition (Matrix)

A matrix is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- We denote this array by a single letter $A$ or by $(a_{ij})$ and
- we say that $A$ has $m$ rows and $n$ columns, or that it is an $m \times n$ matrix.
- The size of $A$ is $m \times n$.
- The number $a_{ij}$ is called the $(i, j)$ entry or scalar.

# Vectors



- An $n \times 1$ matrix is a column vector, or simply a vector:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

  The numbers $v_1, v_2, \ldots$ are known as the components (or entries) of $v$.

- A row vector is a $1 \times n$ matrix
- We write vectors in lower boldcase type (writing by hand we can either underline them or add an arrow over $\mathbf{v}$).

## Definition (Coefficient Matrix)

The matrix $A = (a_{ij})$, whose $(i, j)$ entry is the coefficient $a_{ij}$ of the system of linear equations is called the coefficient matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ then

$$\underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{n \times 1}{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}}$$

# Row operations



How do we find solutions?

$$\begin{array}{l|rcrcrcr}
\texttt{R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\
\texttt{R2:} & 2x_1 & + & x_2 & + & x_3 & = & 4 \\
\texttt{R3:} & x_1 & - & x_2 & + & 2x_3 & = & 5
\end{array}$$

Eliminate one of the variables from two of the equations

$$\begin{array}{l|rcrcrcr}
\texttt{R1'=R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\
\texttt{R2'=R2-2*R1:} & & - & x_2 & - & x_3 & = & -2 \\
\texttt{R3'=R3:} & x_1 & - & x_2 & + & 2x_3 & = & 5
\end{array}$$

$$\begin{array}{l|rcrcrcr}
\texttt{R1'=R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\
\texttt{R2'=R2:} & & - & x_2 & - & x_3 & = & -2 \\
\texttt{R3'=R3-R1:} & & - & 2x_2 & + & x_3 & = & 2
\end{array}$$

We can now eliminate one of the variables in the last two equations to obtain the solution

Row operations that do not alter solutions:

RO1: multiply both sides of an equation by a non-zero constant

RO2: interchange two equations

RO3: add a multiple of one equation to another

These operations only act on the coefficients of the system
For a system $A\mathbf{x} = \mathbf{b}$:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}$$

# Problem Statement



Given the system of linear equations:

$$\begin{array}{l|rcrcrcl} \texttt{R1:} & x_1 & + & x_2 & + & x_3 & = & 3 \\ \texttt{R2:} & 2x_1 & + & x_2 & + & x_3 & = & 4 \\ \texttt{R3:} & x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array}$$

Find whether it has any solution and in case characterize the solutions.

# Augmented Matrix



## Definition (Augmented Matrix and Elementary Row Operations)

For a system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

the augmented matrix of the system is

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

and the elementary row operations are:

- RO1: multiply a row by a non-zero constant
- RO2: interchange two rows
- RO3: add a multiple of one row to another

They modify the linear system into an equivalent system (same solutions)

# Gaussian Elimination: Example

Let's consider the system $A\mathbf{x} = \mathbf{b}$ with:

$$\left[\,A\,\middle|\,\mathbf{b}\,\right] = \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

1. Left most column that is not all zeros
   It is column 1
2. A non-zero entry at the top of this column
   It is the one on the top
3. Make the entry 1
   It is already 1
4. Make all entries below the leading one zero:

$$\begin{array}{l} \texttt{R1'=R1} \\ \texttt{R2'=R2-2R1} \\ \texttt{R3'=R3-R1} \end{array} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & 1 & 2 \end{bmatrix}$$

5. Cover up the top row and apply steps 1. to 4. again

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} & \cancel{3} \\ 0 & -1 & -1 & -2 \\ 0 & -2 & 1 & 2 \end{bmatrix}$$

# Example, cntd. Row Echelon Form

5. Cover up the top row and apply steps 1. to 4. again
1. Left most column that is not all zeros is column 2
2. Non-zero entry at the top of the column
3. Make this entry the leading 1 by elementary row operations RO1 or RO2.
4. Make all entries below the leading 1 zero by RO3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \equiv \begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 3 \\ & & x_2 & + & x_3 & = & 2 \\ & & & & x_3 & = & 2 \end{array}$$

**Definition (Row echelon form)**

A matrix is said to be in row echelon form (or echelon form) if it has the following three properties:

1. the first nonzero entry in each nonzero row is 1
2. a leading 1 in a lower row is further to the right
3. zero rows are at the bottom of the matrix

# Back substitution



$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 3 \\ & & x_2 & + & x_3 & = & 2 \\ & & & & x_3 & = & 2 \end{array}$$

From the row echelon form we solve the system by back substitution:

- from the last equation: set $x_3 = 2$
- substitute $x_3$ in the second equation $\rightsquigarrow x_2$
- substitute $x_2$ and $x_3$ in the first equation $\rightsquigarrow x_1$

# Reduced Row Echelon Form



In the augmented matrix representation:

6. Begin with the last row and add suitable multiples to each row above to get zero above the leading 1.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

**Definition (Reduced row echelon form)**

A matrix is said to be in reduced (row) echelon form if it has the following properties:

1. The matrix is in row echelon form
2. Every column with a leading 1 has zeros elsewhere

From a Reduced Row Echelon Form (RREF) we can read the solution:

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The system has a unique solution.
Is it a correct solution? Let's check:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 3 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

# Gaussian Elimination: Algorithm



Gaussian Elimination algorithm for solving a linear system:
(puts the augmented matrix in a form from which the solution can be read)

1. Find left most column that is not all zeros
2. Get a non-zero entry at the top of this column (pivot element)
3. Make this entry 1 by elementary row operations RO1 or RO2. This entry is called leading one
4. Add suitable multiples of the top row to rows below so that all entries below the leading one become zero
5. Cover up the top row and apply steps (1) and (4) again
   The matrix left is in (row) echelon form
6. Back substitution

# Gauss-Jordan Reduction



Gauss Jordan Reduction algorithm for solving a linear system:
(puts the augmented matrix in a form from which the solution can be read)

1. Find left most column that is not all zeros
2. Get a non-zero entry at the top of this column (pivot element)
3. Make this entry 1 by elementary row operations RO1 or RO2. This entry is called leading one
4. Add suitable multiples of the top row to rows below so that all entries below the leading one become zero
5. Cover up the top row and apply steps (1) and (4) again
   The matrix left is in (row) echelon form
6. Begin with the last row and add suitable multiples to each row above to get zero above the leading 1.
   The matrix left is in reduced (row) echelon form

Will there always be exactly one solution?

$$\begin{array}{l|rcrcl} \texttt{R1:} & & & 2x_3 & = & 3 \\ \texttt{R2:} & 2x_2 & + & 3x_3 & = & 4 \\ \texttt{R3:} & & & x_3 & = & 5 \end{array} \quad \rightarrow \quad \left[\, A \,\middle|\, \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \rightarrow \begin{array}{l} \texttt{R2} \\ \texttt{R1} \\ \phantom{0} \end{array} \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \rightarrow \begin{array}{l} \texttt{R1/2} \\ \phantom{0} \\ \phantom{0} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{array}{l} \phantom{0} \\ \texttt{R3} \\ \texttt{R2} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{array}{l} \phantom{0} \\ \phantom{0} \\ \texttt{R3-2R2} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{array}{l} \phantom{0} \\ \phantom{0} \\ \texttt{-R3/7} \end{array} \begin{bmatrix} 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

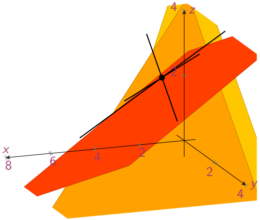
No Solution!
R3: $0 \neq 1$

## Definition (Consistent)

A system of linear equations is said to be consistent if it has at least one solution. It is inconsistent if there are no solutions.

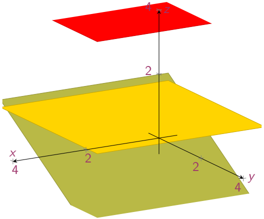
$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}$$

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 2 & 5 \end{array}\right]$$



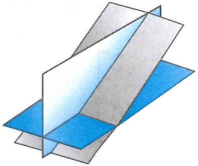

$$\begin{cases} 2x_3 = 3 \\ 2x_2 + 3x_3 = 4 \\ x_3 = 5 \end{cases}$$

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

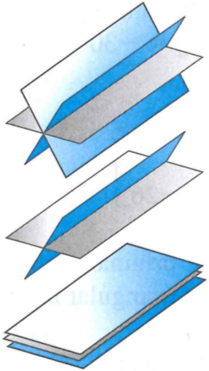


# Geometric Interpretation

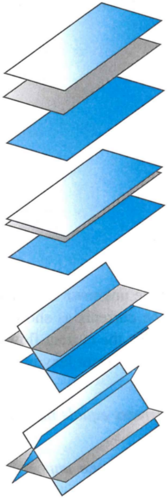
Three equations in three unknowns interpreted as planes in space



Unique solution



Infinitely
many solutions



No solution

### Definition (Overdetermined)

A linear system is said to be over-determined if there are more equations than unknowns. Over-determined systems are usually (but not always) inconsistent.

### Definition (Underdetermined)

A linear system of $m$ equations and $n$ unknowns is said to be under-determined if there are fewer equations than unknowns ($m < n$). They have usually infinitely many solutions (never just one).

# Linear systems with free variables



$$\begin{array}{rcrcrcrcrcr}
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \\
2x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 4 \\
x_1 & - & x_2 & - & x_3 & + & x_4 & + & x_5 & = & 5 \\
x_1 & & & & & + & x_4 & + & x_5 & = & 4
\end{array}$$

$$\left[\, A \,\middle|\, \mathbf{b} \,\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & 1 & 1 & 5 \\ 1 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\rightarrow \quad \begin{array}{r} \\ \texttt{R2-2R1} \\ \texttt{R3-R1} \\ \texttt{R4-R1} \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -1 & 0 & -2 \\ 0 & -2 & -2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{} \quad \begin{matrix} \\ \texttt{(-1)R2} \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & -2 & -2 & 0 & 0 & 2 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{} \quad \begin{matrix} \\ \\ \texttt{R3+2R2} \\ \texttt{R4+R2} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{} \quad \begin{matrix} \\ \\ \texttt{(1/2)R3} \\ \\ \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{} \quad \begin{matrix} \\ \\ \\ \texttt{R4-R3} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row echelon form}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \texttt{R1-R3} \\ \texttt{R2-R3} \\ \rightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \texttt{R1-R2} \\ \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccccccccc} x_1 & + & 0 & + & 0 & + & 0 & + & x_5 & = & 1 \\ & + & x_2 & + & x_3 & + & 0 & + & 0 & = & -1 \\ & & & & & + & x_4 & + & 0 & = & 3 \end{array}$$

$$\begin{array}{ccccccccccc} x_1 & + & 0 & + & 0 & + & 0 & + & x_5 & = & 1 \\ & + & x_2 & + & x_3 & + & 0 & + & 0 & = & -1 \\ & & & & & + & x_4 & + & 0 & = & 3 \end{array}$$

### Definition (Leading variables)

The variables corresponding with leading ones in the reduced row echelon form of an augmented matrix are called leading variables. The other variables are called non-leading variables

- $x_1, x_2$ and $x_4$ are leading variables.
- $x_3, x_5$ are non-leading variables.
- we assign $x_3, x_5$ the arbitrary values $s, t \in \mathbb{R}$ and solve for the leading variables.
- there are infinitely many solutions, represented by the general solution:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1-t \\ -1-s \\ s \\ 3 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

# Homogeneous systems



**Definition (Homogenous system)**

An homogeneous system of linear equations is a linear system of the form $A\mathbf{x} = \mathbf{0}$.

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ is always consistent
  $A\mathbf{0} = \mathbf{0}$.
- If $A\mathbf{x} = \mathbf{0}$ has a unique solution, then it must be the trivial solution $\mathbf{x} = \mathbf{0}$.

In the augmented matrix the last column stays always zero $\rightsquigarrow$ we can omit it.

# Example

$$\begin{array}{rcrcrcrcl} x & + & y & + & 3z & + & w & = & 0 \\ x & - & y & + & z & + & w & = & 0 \\ & & y & + & 2z & + & 2w & = & 0 \end{array} \qquad A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

**Theorem**

*If $A$ is an $m \times n$ matrix with $m < n$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.*

Proof.

- The system is always consistent since homogeneous.
- Matrix $A$ brought in reduced echelon form contains at most $m$ leading ones (variables).
- $n - m \geq 1$ non-leading variables

How about $A\mathbf{x} = \mathbf{b}$ with $A$ $m \times n$ and $m < n$?
If the system is consistent, then there are infinitely many solutions.

# Example

$$\begin{array}{ccccccccc} x & + & y & + & 3z & + & w & = & 2 \\ x & - & y & + & z & + & w & = & 4 \\ & & y & + & 2z & + & 2w & = & 0 \end{array}$$

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$$A\mathbf{x} = \mathbf{0}$$

$$RREF(A)$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$$A\mathbf{x} = \mathbf{b}$$

$$RREF([A|\mathbf{b}])$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

### Definition (Associated homogenous system)

Given a system of linear equations, $A\mathbf{x} = \mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{0}$ is called the associated homogeneous system

Eg:

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

How can you tell from here that $A\mathbf{x} = \mathbf{b}$ is consistent with infinitely many solutions?

### Definition (Null space)

For an $m \times n$ matrix $A$, the null space of $A$ is the subset of $\mathbb{R}^n$ given by

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

where $\mathbf{0} = (0, 0, \ldots, 0)^T$ is the zero vector of $\mathbb{R}^n$

# Summary



- If $A\mathbf{x} = \mathbf{b}$ is consistent, the solutions are of the form:

$$\{\text{solutions of } A\mathbf{x} = \mathbf{b}\} = \mathbf{p} + \{\text{solutions of } A\mathbf{x} = \mathbf{0}\}$$

  - if $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - if $A\mathbf{x} = \mathbf{b}$ has a infinitely many solutions, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions

- $A\mathbf{x} = \mathbf{b}$ may be inconsistent, but $A\mathbf{x} = \mathbf{0}$ is always consistent.