

DM559

Linear and Integer Programming

Lecture 6

## Rank and Range

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# Outline

1. Numerical Issues in Python

2. Rank

3. Range

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## Exercise

Solve the following system of linear equations:

$$\begin{cases} x + z = 1 \\ 3x + 4y + z = 1 \\ + 4y - 2z = -2 \end{cases}$$

$$\det(A) = 1 \begin{vmatrix} 4 & 1 \\ 4 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix} = 0$$

Hence we cannot solve by inverse nor by Cramer's rule. We proceed by Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 0 & 4 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x &= 1 - t \\ y &= -\frac{1}{2} + \frac{1}{2}t, \forall t \in \mathbb{R} \\ z &= t \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} t, \forall t \in \mathbb{R} \quad \text{infinitely many solutions}$$

In Python:

```
import numpy as np
r1=np.array([1,0,1])
r2=np.array([3,4,1])
r3=r2-3*r1
A=np.vstack([r1,r2,r3])
```

```
np.linalg.det(A)
# 1.3322676295501906e-15
np.dot(np.linalg.inv(A), [1,1,-2])
# array([-1., 0., 1.])
np.dot(np.linalg.inv(A),A)
# array([[ 0., 0., -1.],[ 0., 1., 0.],[ 0., 0., 1.]])
```

```
np.linalg.solve(A, [1,1,-2])
# array([ 0., 0., 1.])
```

Hence, Python for numerical reasons does not recognize the determinant to be null and solves the system returning only one particular solution.

Knowledge of the theory of linear algebra is important to avoid mistakes!

```
import sympy as sy
M=sy.Matrix(A)
M.rref()
```

or you can carry out the row operations yourself

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# Rank

- Synthesis of what we have seen so far under the light of two new concepts: **rank** and **range** of a matrix
- We saw that:  
every matrix is row-equivalent to a matrix in reduced row echelon form.

## Definition (Rank of Matrix)

The **rank** of a matrix  $A$ ,  $\text{rank}(A)$ , is

- the number of non-zero rows, or equivalently
- the number of leading ones

in a row echelon matrix obtained from  $A$  by elementary row operations.

↪ For an  $m \times n$  matrix  $A$ ,

$$\text{rank } A \leq \min\{m, n\},$$

where  $\min\{m, n\}$  denotes the smaller of the two integers  $m$  and  $n$ .

## Example

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 6 \end{bmatrix} \xrightarrow{\substack{R'_2 = R_2 - 2R_1 \\ R'_3 = R_3 - 3R_1}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \end{bmatrix} \xrightarrow{\substack{R'_2 = -R_2 \\ R'_3 = R_3 - R_2}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \text{rank}(M) = 2$$



# Extension of the main theorem

## Theorem

If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent:

1.  $A$  is invertible
2.  $A\mathbf{x} = \mathbf{b}$  has a unique solution for any  $\mathbf{b} \in \mathbb{R}^n$
3.  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution,  $\mathbf{x} = \mathbf{0}$
4. the reduced row echelon form of  $A$  is  $I$ .
5.  $|A| \neq 0$
6. the rank of  $A$  is  $n$

# Rank and Systems of Linear Equations

$$x + 2y + z = 1$$

$$2x + 3y = 5$$

$$3x + 5y + z = 4$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 4 \end{array} \right] \xrightarrow{\substack{R'_2=R_2-2R_1 \\ R'_3=R_3-3R_1}} \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 1 \end{array} \right] \xrightarrow{\substack{R'_2=-R_2 \\ R'_3=R_3-R_2}} \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -2 \end{array} \right]$$

$$x + 2y + z = 1$$

$$x + 2z = -3$$

$$0x + 0y + 0z = -2$$

It is inconsistent!

The last row is of the type  $0 = a$ ,  $a \neq 0$ , that is, the augmenting matrix has a leading one in the last column

$$\text{rank}(A) = 2 \neq \text{rank}(A | \mathbf{b}) = 3$$

1. A system  $A\mathbf{x} = \mathbf{b}$  is consistent if and only if the rank of the augmented matrix is precisely the same as the rank of the matrix  $A$ .

2. If an  $m \times n$  matrix  $A$  has rank  $m$ , the system of linear equations,  $A\mathbf{x} = \mathbf{b}$ , will be consistent for all  $\mathbf{b} \in \mathbb{R}^n$
- Since  $A$  has rank  $m$  then there is a leading one in every row. Hence  $[A \mid \mathbf{b}]$  cannot have a row  $[0, 0, \dots, 0, 1] \implies \text{rank } A \neq \text{rank}(A \mid \mathbf{b})$
  - $[A \mid \mathbf{b}]$  has also  $m$  rows  $\implies \text{rank}(A) \neq \text{rank}(A \mid \mathbf{b})$
  - Hence,  $\text{rank}(A) = \text{rank}(A \mid \mathbf{b})$

### Example

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 0 & 5 \\ 3 & 5 & 1 & 4 \end{bmatrix} \rightarrow \dots \rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{rank}(B) = 3$$

Any system  $B\mathbf{x} = \mathbf{d}$  in 4 unknowns and 3 equalities with  $\mathbf{d} \in \mathbb{R}^3$  is consistent.

Since  $\text{rank}(A)$  is smaller than the number of variables, then there is a **non-leading variable**. Hence infinitely many solutions!

## Example

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \dots \rightarrow \begin{bmatrix} 1 & 3 & 0 & 4 & 0 & -28 \\ 0 & 0 & 1 & 2 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}([A|\mathbf{b}]) = 3 < 5 = n$$

$$\begin{aligned} x_1 + 3x_2 + 4x_4 &= -28 \\ x_3 + 2x_4 &= -14 \\ x_5 &= 5 \end{aligned}$$

$x_1, x_3, x_5$  are leading variables;  $x_2, x_4$  are non-leading variables (set them to  $s, t \in \mathbb{R}$ )

$$\begin{aligned} x_1 &= -28 - 3s - 4t \\ x_2 &= s \\ x_3 &= -14 - 2t \\ x_4 &= t \\ x_5 &= 5 \end{aligned} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -28 \\ 0 \\ -14 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} t$$

# Summary

Let  $A\mathbf{x} = \mathbf{b}$  be a general linear system in  $n$  variables and  $m$  equations:

- If  $\text{rank}(A) = r < m$  and  $\text{rank}(A | \mathbf{b}) = r + 1$  then the system is **inconsistent**. (the row echelon form of the augmented matrix has a row  $[0 \ 0 \ \dots \ 0 \ 1]$ )
- If  $\text{rank}(A) = r = \text{rank}(A | \mathbf{b})$  then the system is consistent and there are  $n - r$  free variables; if  $r < n$  there are **infinitely many solutions**, if  $r = n$  there are no free variables and the **solution is unique**

Let  $A\mathbf{x} = \mathbf{0}$  be an homogeneous system in  $n$  variables and  $m$  equations,  $\text{rank}(A) = r$  (always consistent):

- if  $r < n$  there are **infinitely many solutions**, if  $r = n$  there are no free variables and the **solution is unique**,  $\mathbf{x} = \mathbf{0}$ .

# General solutions in vector notation

## Example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -28 \\ 0 \\ -14 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} t, \quad \forall s, t \in \mathbb{R}$$

For  $A\mathbf{x} = \mathbf{b}$ :

$$\mathbf{x} = \mathbf{p} + \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-r} \mathbf{v}_{n-r}, \quad \forall \alpha_i \in \mathbb{R}, i = 1, \dots, n-r$$

Note:

- if  $\alpha_i = 0, \forall i = 1, \dots, n-r$  then  $A\mathbf{p} = \mathbf{b}$ , ie,  $\mathbf{p}$  is a particular solution
- if  $\alpha_1 = 1$  and  $\alpha_i = 0, \forall i = 2, \dots, n-r$  then

$$A(\mathbf{p} + \mathbf{v}_1) = \mathbf{b} \rightarrow A\mathbf{p} + A\mathbf{v}_1 = \mathbf{b} \xrightarrow{A\mathbf{p}=\mathbf{b}} A\mathbf{v}_1 = \mathbf{0}$$

Thus (recall that  $\mathbf{x} = \mathbf{p} + \mathbf{z}$ ,  $\mathbf{z} \in N(A)$ ):

- If  $A$  is an  $m \times n$  matrix of rank  $r$ , the general solutions of  $A\mathbf{x} = \mathbf{b}$  is the sum of:
  - a particular solution  $\mathbf{p}$  of the system  $A\mathbf{x} = \mathbf{b}$  and
  - a linear combination  $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{n-r}\mathbf{v}_{n-r}$  of solutions  $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{n-r}$  of the homogeneous system  $A\mathbf{x} = \mathbf{0}$
- If  $A$  has rank  $n$ , then  $A\mathbf{x} = \mathbf{0}$  only has the solution  $\mathbf{x} = \mathbf{0}$  and so  $A\mathbf{x} = \mathbf{b}$  has a unique solution:  
 $\mathbf{p}$

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# Range

## Definition (Range of a matrix)

Let  $A$  be an  $m \times n$  matrix, the **range** of  $A$ , denoted by  $R(A)$ , is the subset of  $\mathbb{R}^m$  given by

$$R(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

That is, the range is the set of all vectors  $\mathbf{y} \in \mathbb{R}^m$  of the form  $\mathbf{y} = A\mathbf{x}$  for some  $\mathbf{x} \in \mathbb{R}^n$ , or all  $\mathbf{y} \in \mathbb{R}^m$  for which the system  $A\mathbf{x} = \mathbf{y}$  is consistent.

Recall, if  $\mathbf{x} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$  is any vector in  $\mathbb{R}^n$  and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \quad i = 1, \dots, n.$$

Then  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$  and

$$A\mathbf{x} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \dots + \alpha_n\mathbf{a}_n$$

that is, vector  $A\mathbf{x}$  in  $\mathbb{R}^n$  as a **linear combination** of the column vectors of  $A$

Hence  $R(A)$  is the set of all **linear combinations** of the columns of  $A$ .

$\rightsquigarrow$  the range is also called the **column space** of  $A$ :

$$R(A) = \{\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \dots + \alpha_n\mathbf{a}_n \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\}$$

Thus,  $A\mathbf{x} = \mathbf{b}$  is consistent iff  $\mathbf{b}$  is in the range of  $A$ , ie, a linear combination of the columns of  $A$

## Example

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

Then, for  $\mathbf{x} = [\alpha_1, \alpha_2]^T$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \alpha_1 + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \alpha_2$$

so

$$R(A) = \left\{ \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ -\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

## Example

$$\begin{cases} x + 2y = 0 \\ -x + 3y = -5 \\ 2x + y = 3 \end{cases}$$

$$Ax = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2\mathbf{a}_1 - \mathbf{a}_2$$

$$\begin{cases} x + 2y = 1 \\ -x + 3y = -5 \\ 2x + y = 2 \end{cases}$$

$$Ax = \mathbf{0}$$

has only the trivial solution  $\mathbf{x} = \mathbf{0}$ . (Why?) Only way:

$$0 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0\mathbf{a}_1 + 0\mathbf{a}_2 = \mathbf{0}$$

Hence no way to express  $[1, -5, 2]$  as linear expression of the two columns of  $A$ .