DM559

Linear and Integer Programming

Lecture 7
Vector Spaces
Linear Independence, Bases and Dimension

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

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Premise

- We move to a higher level of abstraction
- A vector space is a set with an addition and scalar multiplication that behave appropriately, that is, like $\mathbb{R}^n$
- Imagine a vector space as a class of a generic type (template) in object oriented programming, equipped with two operations.

Vector Spaces

Definition (Vector Space)

A (real) vector space V is a non-empty set equipped with an addition and a scalar multiplication operation such that for all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$:

1. $\mathbf{u} + \mathbf{v} \in V$ (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative law for addition)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associative law for addition)
4. there is a single member $\mathbf{0}$ of V, called the zero vector, such that for all $\mathbf{v} \in V, \mathbf{v} + \mathbf{0} = \mathbf{v}$
5. for every $\mathbf{v} \in V$ there is an element $\mathbf{w} \in V$, written $-\mathbf{v}$, called the negative of $\mathbf{v}$, such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
6. $\alpha \mathbf{v} \in V$ (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ (distributive law)
8. $(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$ (distributive law)
9. $\alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v}$ (associative law for vector multiplication)
10. $1\mathbf{v} = \mathbf{v}$

Examples

- set $\mathbb{R}^n$
- but the set of objects for which the vector space defined is valid are more than the vectors in $\mathbb{R}^n$.
- set of all functions $F : \mathbb{R} \to \mathbb{R}$.
  We can define an addition $f + g$:
  $$(f + g)(x) = f(x) + g(x)$$
  and a scalar multiplication αf:
  $$(\alpha f)(x) = \alpha f(x)$$
- Example: $x + x^2$ and $2x$. They can represent the result of the two operations.
- What is $-f$? and the zero vector?

The axioms given are minimum number needed.
Other properties can be derived:
For example:

$$(-1)\mathbf{x} = -\mathbf{x}$$

Proof:

$$\mathbf{0} = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = \mathbf{x} + (-1)\mathbf{x}$$

Adding $-\mathbf{x}$ on both sides:

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + \mathbf{x} + (-1)\mathbf{x} = (-1)\mathbf{x}$$

which proves that $-\mathbf{x} = (-1)\mathbf{x}$.

Try the same with $-f$.

Examples

- $V = \{\mathbf{0}\}$

- the set of all $m \times n$ matrices

- the set of all infinite sequences of real numbers, $\mathbf{y} = \{y_1, y_2, \ldots, y_n, \ldots,\}, y_i \in \mathbb{R}$.
  ($\mathbf{y} = \{y_n\}$, $n \geq 1$)
  - addition of $\mathbf{y} = \{y_1, y_2, \ldots, y_n, \ldots,\}$ and $\mathbf{z} = \{z_1, z_2, \ldots, z_n, \ldots,\}$ then:

    $$\mathbf{y} + \mathbf{z} = \{y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n, \ldots,\}$$
  - multiplication by a scalar $\alpha \in \mathbb{R}$:

    $$\alpha\mathbf{y} = \{\alpha y_1, \alpha y_2, \ldots, \alpha y_n, \ldots,\}$$

- set of all vectors in $\mathbb{R}^3$ with the third entry equal to 0 (verify closure):

  $$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| \; x, y \in \mathbb{R} \right\}$$

Linear Combinations

Definition (Linear Combination)

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V, the vector

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k$$

is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
The scalars α_i are called coefficients.

- To find the coefficients that given a set of vertices express by linear combination a given vector, we solve a system of linear equations.
- If F is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ then the function $f : x \mapsto 2x^2 + 3x + 4$ can be expressed as a linear combination of:
  $g : x \mapsto x^2$, $h : x \mapsto x$, $k : x \mapsto 1$ that is:
  $$f = 2g + 3h + 4k$$
- Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, is it possible to represent any point in the Cartesian plane?

Subspaces

Definition (Subspace)

A subspace W of a vector space V is a non-empty subset of V that is itself a vector space under the same operations of addition and scalar multiplication as V.

Theorem

Let V be a vector space. Then a non-empty subset W of V is a subspace if and only if both the following hold:

- *for all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$*
  (W is closed under addition)
- *for all $\mathbf{v} \in W$ and $\alpha \in \mathbb{R}$, $\alpha\mathbf{v} \in W$*
  (W is closed under scalar multiplication)

ie, all other axioms can be derived to hold true

Example

- The set of all vectors in $\mathbb{R}^3$ with the third entry equal to 0.
- The set $\{\mathbf{0}\}$ is not empty, it is a subspace since $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $\alpha\mathbf{0}=\mathbf{0}$ for any $\alpha\in\mathbb{R}$.

Example

In $\mathbb{R}^2$, the lines $y=2x$ and $y=2x+1$ can be defined as the sets of vectors:

$$S=\left\{\begin{bmatrix}x\\y\end{bmatrix}\;\middle|\; y=2x, x\in\mathbb{R}\right\}\qquad U=\left\{\begin{bmatrix}x\\y\end{bmatrix}\;\middle|\; y=2x+1, x\in\mathbb{R}\right\}$$

$$S=\{\mathbf{x}\mid \mathbf{x}=t\mathbf{v}, t\in\mathbb{R}\}\qquad U=\{\mathbf{x}\mid \mathbf{x}=\mathbf{p}+t\mathbf{v}, t\in\mathbb{R}\}$$

$$\mathbf{v}=\begin{bmatrix}1\\2\end{bmatrix},\ \mathbf{p}=\begin{bmatrix}0\\1\end{bmatrix}$$

Example (cntd)

1. The set S is non-empty, since $\mathbf{0} = 0\mathbf{v} \in S$.
2. closure under addition:

$$\mathbf{u} = s\begin{bmatrix}1\\2\end{bmatrix} \in S, \quad \mathbf{w} = t\begin{bmatrix}1\\2\end{bmatrix} \in S, \qquad \text{for some } s, t \in \mathbb{R}$$

$\mathbf{u} + \mathbf{w} = s\mathbf{v} + t\mathbf{v} = (s+t)\mathbf{v} \in S$ since $s + t \in \mathbb{R}$

3. closure under scalar multiplication:

$$\mathbf{u} = s\begin{bmatrix}1\\2\end{bmatrix} \in S \qquad \text{for some } s \in \mathbb{R}, \qquad \alpha \in \mathbb{R}$$

$\alpha\mathbf{u} = \alpha(s(\mathbf{v})) = (\alpha s)\mathbf{v} \in S$ since $\alpha s \in \mathbb{R}$

Note that:

- $\mathbf{u}, \mathbf{w}$ and $\alpha \in \mathbb{R}$ must be arbitrary

Example (cntd)

1. $\mathbf{0} \notin U$
2. U is not closed under addition:

$$\begin{bmatrix}0\\1\end{bmatrix} \in U, \begin{bmatrix}1\\3\end{bmatrix} \in U \qquad \text{but} \qquad \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix} \notin U$$

3. U is not closed under scalar multiplication

$$\begin{bmatrix}0\\1\end{bmatrix} \in U, 2 \in \mathbb{R} \qquad \text{but} \qquad 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \notin U$$

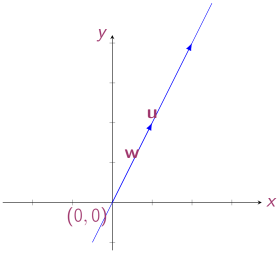
Note that:

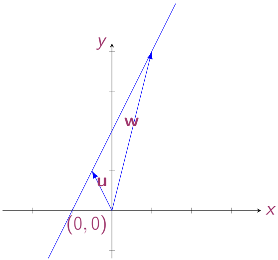
- proving just one of the above couterexamples is enough to show that U is not a subspace
- it is sufficient to make them fail for particular choices
- a good place to start is checking whether $\mathbf{0} \in S$. If not then S is not a subspace

Theorem

A non-empty subset W of a vector space is a subspace if and only if for all $\mathbf{u}, \mathbf{v} \in W$ and all $\alpha, \beta \in \mathbb{R}$, we have $\alpha\mathbf{u} + \beta\mathbf{v} \in W$.
That is, W is closed under linear combination.

Geometric interpretation:





⇝ The line $y = 2x + 1$ is an affine subset, a „translation“ of a subspace

Null space of a Matrix is a Subspace

Theorem

For any $m \times n$ matrix A, $N(A)$, ie, the solutions of $A\mathbf{x} = \mathbf{0}$, is a **subspace** *of* $\mathbb{R}^n$

Proof

1. $A\mathbf{0} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{0} \in N(A)$

2. Suppose $\mathbf{u}, \mathbf{v} \in N(A)$, then $\mathbf{u} + \mathbf{v} \in N(A)$:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

3. Suppose $\mathbf{u} \in N(A)$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{u} \in N(A)$:

$$A(\alpha\mathbf{u}) = A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha\mathbf{0} = \mathbf{0}$$

□

The set of solutions S to a general system $A\mathbf{x} = \mathbf{b}$ is not a **subspace** of $\mathbb{R}^n$ because $\mathbf{0} \notin S$

Affine subsets

Definition (Affine subset)

If W is a **subspace** of a **vector space** V and $\mathbf{x} \in V$, then the set $\mathbf{x} + W$ defined by

$$\mathbf{x} + W = \{\mathbf{x} + \mathbf{w} \mid \mathbf{w} \in W\}$$

is said to be an affine subset of V.

The set of solutions S to a general system $A\mathbf{x} = \mathbf{b}$ is an affine **subspace**, indeed recall that if $\mathbf{x}_0$ is any solution of the system

$$S = \{\mathbf{x}_0 + \mathbf{z} \mid \mathbf{z} \in N(A)\}$$

Range of a Matrix is a Subspace

Theorem

For any $m \times n$ matrix A, $R(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$ is a **subspace** *of $\mathbb{R}^m$*

Proof

1. $A\mathbf{0} = \mathbf{0} \implies \mathbf{0} \in R(A)$
2. Suppose $\mathbf{u}, \mathbf{v} \in R(A)$, then $\mathbf{u} + \mathbf{v} \in R(A)$:
   ...
3. Suppose $\mathbf{u} \in R(A)$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{u} \in R(A)$:
   ...

Linear Span

- If $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_k\mathbf{v}_k$, then $\mathbf{v} + \mathbf{w}$ and $s\mathbf{v}, s \in \mathbb{R}$ are also linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.
- The set of all linear combinations of a given set of vectors of a **vector space** V forms a **subspace**:

Definition (Linear span)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. The linear span of $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, denoted by $\mathrm{Lin}(X)$, that is:

$$\mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}$$

Theorem

If $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors of a vector space V, then $\mathrm{Lin}(X)$ is a **subspace** *of V and is also called the* **subspace** *spanned by X.*
It is the smallest **subspace** *containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.*

Example

- $\mathrm{Lin}(\{\mathbf{v}\}) = \{\alpha\mathbf{v} \mid \alpha \in \mathbb{R}\}$ defines a line in $\mathbb{R}^n$.
- Recall that a plane in $\mathbb{R}^3$ has two equivalent representations:

$$ax + by + cz = d \qquad \text{and} \qquad \mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

  where $\mathbf{v}$ and $\mathbf{w}$ are non parallel.

  - If $d = 0$ and $\mathbf{p} = \mathbf{0}$, then

$$\{\mathbf{x} \mid \mathbf{x} = s\mathbf{v} + t\mathbf{w}, s, t, \in \mathbb{R}\} = \mathrm{Lin}(\{\mathbf{v}, \mathbf{w}\})$$

    and hence a **subspace** of $\mathbb{R}^n$.

  - If $d \neq 0$, then the plane is not a **subspace**. It is an affine subset, a translation of a **subspace**.

    (recall that one can also show directly that a subset is a **subspace** or not)

Spanning Sets of a Matrix

Definition (Column space)

If A is an $m \times n$ matrix, and if $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ denote the columns of A, then the column space or range of A is

$$CS(A) = R(A) = \text{Lin}(\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\})$$

and is a **subspace** of $\mathbb{R}^m$.

Definition (Row space)

If A is an $m \times n$ matrix, and if $\overrightarrow{\mathbf{a}}_1, \overrightarrow{\mathbf{a}}_2, \dots, \overrightarrow{\mathbf{a}}_k$ denote the rows of A, then the row space of A is

$$RS(A) = \text{Lin}(\{\overrightarrow{\mathbf{a}}_1, \overrightarrow{\mathbf{a}}_2, \dots, \overrightarrow{\mathbf{a}}_k\})$$

and is a **subspace** of $\mathbb{R}^n$.

- If A is an $m \times n$ matrix, then for any $\mathbf{r} \in RS(A)$ and any $\mathbf{x} \in N(A)$, $\langle \mathbf{r}, \mathbf{x} \rangle = 0$; that is, $\mathbf{r}$ and $\mathbf{x}$ are orthogonal, $RS(A) \perp N(A)$. (hint: look at $A\mathbf{x} = \mathbf{0}$)

Summary

We have seen:

- Definition of **vector space** and **subspace**
- Linear combinations as the main way to work with vector spaces
- Proofs that a given set is a vector space
- Proofs that a given subset of a vector space is a **subspace** or not
- Definition of linear span of set of vectors
- Definition of row and column spaces of a matrix
  $CS(A) = R(A)$ and $RS(A) \perp N(A)$

Outline

1. Vector Spaces and Subspaces

2. Linear independence

3. Bases and Dimension

Linear Independence

Definition (Linear Independence)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent (or form a linearly independent set) if and only if the vector equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

has the unique solution

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Definition (Linear Dependence)

Let V be a **vector space** and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent (or form a linearly dependent set) if and only if there are real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

Example

In $\mathbb{R}^2$, the vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are linearly independent. Indeed:

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{cases} \alpha + \beta = 0 \\ 2\alpha - \beta = 0 \end{cases}$$

The homogeneous linear system has only the trivial solution, $\alpha = 0, \beta = 0$, so linear independence.

Example

In $\mathbb{R}^3$, the following vectors are linearly dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

Indeed: $2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$

Theorem

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$ *is linearly dependent if and only if at least one vector* $\mathbf{v}_i$ *is a linear combination of the other vectors.*

Proof

$\Longrightarrow$

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are linearly dependent then

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}$$

has a solution with some $\alpha_i \neq 0$, then:

$$\mathbf{v}_i = -\frac{\alpha_1}{\alpha_i}\mathbf{v}_1 - \frac{\alpha_2}{\alpha_i}\mathbf{v}_2 - \cdots - \frac{\alpha_{i-1}}{\alpha_i}\mathbf{v}_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}\mathbf{v}_{i+1} + \cdots - \frac{\alpha_k}{\alpha_i}\mathbf{v}_k$$

which is a linear combination of the other vectors

$\Longleftarrow$

If $\mathbf{v}_i$ is a lin combination of the other vectors, eg,

$$\mathbf{v}_i = \beta_1\mathbf{v}_1 + \cdots + \beta_{i-1}\mathbf{v}_{i-1} + \beta_{i+1}\mathbf{v}_{i+1} + \cdots + \beta_k\mathbf{v}_k$$

then

$$\beta_1\mathbf{v}_1 + \cdots + \beta_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + \beta_{i+1}\mathbf{v}_{i+1} + \cdots + \beta_k\mathbf{v}_k = \mathbf{0}$$

□

Corollary

Two vectors are linearly dependent if and only if at least one vector is a scalar multiple of the other.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

are linearly independent

Theorem

In a **vector space** V*, a non-empty set of vectors that contains the zero vector is linearly dependent.*

Proof:

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subset V$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{0}\}$$

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_k + a\mathbf{0} = \mathbf{0}, \qquad a \neq 0$$

Uniqueness of linear combinations

Theorem

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent vectors in V and if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_k\mathbf{v}_k$$

then

$$a_1 = b_1, \quad a_2 = b_2, \quad \ldots \quad a_k = b_k.$$

- If a vector $\mathbf{x}$ can be expressed as a linear combination of linearly independent vectors, then this can be done in only one way

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

Testing for Linear Independence in $\mathbb{R}^n$

For k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$$

is equivalent to

$$A\mathbf{x}$$

where A is the $n \times k$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and $\mathbf{x} = [\alpha_1, \alpha_2, \ldots, \alpha_k]^T$:

Theorem

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly dependent if and only if the linear system $A\mathbf{x} = \mathbf{0}$, where A is the matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_k]$, has a solution other than $\mathbf{x} = \mathbf{0}$.
Equivalently, the vectors are linearly independent precisely when the only solution to the system is $\mathbf{x} = \mathbf{0}$.

If vectors are linearly dependent, then any solution $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} = [\alpha_1, \alpha_2 \ldots, \alpha_k]^T$ of $A\mathbf{x} = \mathbf{0}$ gives a non-trivial linear combination $A\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k = \mathbf{0}$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

are linearly dependent.
We solve $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -5 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

The general solution is

$$\mathbf{v} = \begin{bmatrix} t \\ -3t \\ t \end{bmatrix}$$

and $A\mathbf{x} = t\mathbf{v}_1 - 3t\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$

Hence, for $t = 1$ we have: $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Recall that $A\mathbf{x} = \mathbf{0}$ has precisely one solution $\mathbf{x} = \mathbf{0}$ iff the $n \times k$ matrix is row equiv. to a row echelon matrix with k leading ones, ie, iff $\text{rank}(A) = k$

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent iff the $n \times k$ matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_k]$ has rank k.

Theorem

The maximum size of a linearly independent set of vectors in $\mathbb{R}^n$ is n.

- $\text{rank}(A) \leq \min\{n, k\}$, hence $\text{rank}(A) \leq n \Rightarrow$ when lin. indep. $k \leq n$.
- we exhibit an example that has exactly n independent vectors in $\mathbb{R}^n$ (there are infinite examples):

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This is known as the standard basis of $\mathbb{R}^n$.

Example

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 1 \end{bmatrix} \right\} \quad \text{lin. dep. since } 5 > n = 4$$

$$L_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix} \right\} \quad \text{lin. indep.}$$

$$L_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\} \quad \text{lin. dep. since } \operatorname{rank}(A) = 2$$

$$L_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{lin. dep. since } L_3 \subseteq L_4$$

Linear Independence and Span in $\mathbb{R}^n$

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.
What are the conditions for S to span $\mathbb{R}^n$ and be linearly independent?

Let A be the $n \times k$ matrix whose columns are the vectors from S.

- S spans $\mathbb{R}^n$ if for any $v \in \mathbb{R}^n$ the linear system $A\mathbf{x} = \mathbf{v}$ is consistent for all $\mathbf{v} \in \mathbb{R}^n$. This happens when $\text{rank}(A) = n$, hence $k \geq n$
- S is linearly independent iff the linear system $A\mathbf{x} = \mathbf{0}$ has a unique solution. This happens when $\text{rank}(A) = k$, Hence $k \leq n$

Hence, to span $\mathbb{R}^n$ and to be linearly independent, the set S must have exactly n vectors and the square matrix A must have $\det(A) \neq 0$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 5 & 1 \end{vmatrix} = 30 \neq 0$$

Outline

Bases

Definition (Basis)

Let V be a **vector space**. Then the subset $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of V is said to be a basis for V if:

1. B is a linearly independent set of vectors, and
2. B spans V; that is, $V = \mathrm{Lin}(B)$

Theorem

*If V is a **vector space**, then a smallest spanning set is a basis of V.*

Theorem

$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V if and only if any $\mathbf{v} \in V$ is a unique linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Example

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.
the vectors are linearly independent and for any $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$,
$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$, ie,

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example

The set below is a basis of $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

- any vector $\mathbf{b} \in \mathbb{R}^2$ is a linear combination of the two vectors in S
  $\rightsquigarrow A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$.
- S spans $\mathbb{R}^2$ and is linearly independent

Example

Find a basis of the **subspace** of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x + y - 3z = 0 \right\}.$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x + 3z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = x\mathbf{v} + z\mathbf{w}, \quad \forall x, z \in \mathbb{R}$$

The set $\{\mathbf{v}, \mathbf{w}\}$ spans W. The set is also independent:

$$\alpha\mathbf{v} + \beta\mathbf{w} = \mathbf{0} \implies \alpha = 0, \beta = 0$$

Coordinates

Definition (Coordinates)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V, then any vector $\mathbf{v} \in V$ can be expressed uniquely as $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ then the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coordinates of $\mathbf{v}$ with respect to the basis S.

We use the notation

$$[\mathbf{v}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_S$$

to denote the coordinate vector of $\mathbf{v}$ in the basis S.

- We assume the order of the vectors in the basis to be fixed: aka, ordered basis
- Note that $[\mathbf{v}]_S$ is a vector in $\mathbb{R}^n$: Coordinate mapping creates a one-to-one correspondence between a general vector space V and the fmailiar vector space $\mathbb{R}^n$.

Example

Consider the two basis of $\mathbb{R}^2$:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}_B \qquad [\mathbf{v}]_S = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_S$$

In the standard basis the coordinates of $\mathbf{v}$ are precisely the components of the vector $\mathbf{v}$.
In the basis S, they are such that

$$\mathbf{v} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Extension of the main theorem

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. *A is invertible*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}$*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$*
4. *the reduced row echelon form of A is I.*
5. $|A| \neq 0$
6. *The rank of A is n*
7. *The column vectors of A are a basis of $\mathbb{R}^n$*
8. *The rows of A (written as vectors) are a basis of $\mathbb{R}^n$*

(The last statement derives from $|A^T| = |A|$.)
Hence, simply calculating the determinant can inform on all the above facts.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

This set is linearly dependent since $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$
so $\mathbf{v}_3 \in \mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\})$ and $\mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \mathrm{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$.
The linear span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^3$ is a plane:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2 = s\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The vector $\mathbf{x}$ belongs to the **subspace** iff it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, that is, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}$ are linearly dependent or:

$$|A| = \begin{vmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 3 & 5 & z \end{vmatrix} = 0 \qquad \Longrightarrow \qquad |A| = 7x + y - 3z = 0$$

Dimension

Theorem

Let V be a **vector space** *with a basis*

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

of n vectors. Then any set of $n+1$ vectors is linearly dependent.

Proof:
Omitted (choose an arbitrary set of $n+1$ vectors in V and show that since any of them is spanned by the basis then the set must be linearly dependent.)

It follows that:

Theorem

Let a **vector space** V *have a finite basis consisting of* r *vectors. Then any basis of* V *consists of exactly* r *vectors.*

Definition (Dimension)

The number of k vectors in a finite basis of a **vector space** V is the dimension of V and is denoted by $\dim(V)$.
The **vector space** $V = \{\mathbf{0}\}$ is defined to have dimension 0.

- a plane in $\mathbb{R}^2$ is a two-dimensional **subspace**
- a line in $\mathbb{R}^n$ is a one-dimensional **subspace**
- a hyperplane in $\mathbb{R}^n$ is an $(n-1)$-dimensional **subspace** of $\mathbb{R}^n$
- the **vector space** F of real functions is an infinite-dimensional **vector space**
- the **vector space** of real-valued sequences is an infinite-dimensional **vector space**.

Dimension and bases of Subspaces

Example

The plane W in $\mathbb{R}^3$

$$W = \{\mathbf{x} \mid x + y - 3z = 0\}$$

has a basis consisting of the vectors $\mathbf{v}_1 = [1, 2, 1]^T$ and $\mathbf{v}_2 = [3, 0, 1]^T$.

Let $\mathbf{v}_3$ be any vector $\notin W$, eg, $\mathbf{v}_3 = [1, 0, 0]^T$. Then the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

Basis of a Linear Space

If we are given k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, how can we find a basis for $\text{Lin}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\})$?

We can:

- create an $n \times k$ matrix (vectors as columns) and find a basis for the column space by putting the matrix in reduced row echelon form

Definition (Rank and nullity)

The rank of a matrix A is

$$\text{rank}(A) = \dim(R(A))$$

The nullity of a matrix A is

$$\text{nullity}(A) = \dim(N(A))$$

Although **subspaces** of possibly different Euclidean spaces:

Theorem

If A is an $m \times n$ matrix, then

$$\dim(RS(A)) = \dim(CS(A)) = \text{rank}(A)$$

Theorem (Rank-nullity theorem)

For an $m \times n$ matrix A

$$\text{rank}(A) + \text{nullity}(A) = n \qquad (\dim(R(A)) + \dim(N(A)) = n)$$

Summary

- Linear dependence and independence
- Determine linear dependency of a set of vectors, ie, find non-trivial lin. combination that equal zero
- Basis
- Find a basis for a linear space
- Dimension (finite, infinite)