DM559

Linear and Integer Programming

Lecture 8
Change of Basis

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Resume

- Linear dependence and independence
- Determine linear dependency of a set of vectors, ie, find non-trivial lin. combination that equal zero
- Basis
- Find a basis for a linear space
- Dimension (finite, infinite)

Outline

Coordinates

Recall:

Definition (Coordinates)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V, then

- any vector $\mathbf{v} \in V$ can be expressed uniquely as $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$
- and the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coordinates of $\mathbf{v}$ wrt the basis S.

To denote the coordinate vector of $\mathbf{v}$ in the basis S we use the notation

$$[\mathbf{v}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_S$$

- In the standard basis the coordinates of $\mathbf{v}$ are precisely the components of the vector $\mathbf{v}$: $\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n$
- How to find coordinates of a vector $\mathbf{v}$ wrt another basis?

Transition from Standard to Basis B

Definition (Transition Matrix)

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$. The coordinates of a vector $\mathbf{x}$ wrt B, $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T = [\mathbf{x}]_B$, are found by solving the linear system:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = \mathbf{x} \qquad \text{that is} \qquad [\mathbf{v}_1\ \mathbf{v}_2\ \cdots \mathbf{v}_n][\mathbf{x}]_B = \mathbf{x}$$

We call P the matrix whose columns are the basis vectors:

$$P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots \mathbf{v}_n]$$

Then for any vector $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{x} = P[\mathbf{x}]_B$$ transition matrix from B coords to standard coords

moreover P is invertible (columns are a basis):

$$[\mathbf{x}]_B = P^{-1}\mathbf{x}$$ transition matrix from standard coords to B coords

Example

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \qquad [\mathbf{v}]_B = \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$\det(P) = 4 \neq 0$ so B is a basis of $\mathbb{R}^3$

We derive the standard coordinates of $\mathbf{v}$:

$$\mathbf{v} = 4 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - 5 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \\ -5 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}_B = \begin{bmatrix} -9 \\ -3 \\ -5 \end{bmatrix}$$

Example (cntd)

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}, \qquad [\mathbf{x}]_S = \begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix}$$

We derive the B coordinates of vector $\mathbf{x}$:

$$\begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

either we solve $P\mathbf{a} = \mathbf{x}$ in $\mathbf{a}$ by Gaussian elimination or we find the inverse P^{-1}:

$$[\mathbf{x}]_B = P^{-1}\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}_B \qquad \text{check the calculation}$$

What are the B coordinates of the basis vector? ($[1, 0, 0], [0, 1, 0], [0, 0, 1]$)

Change of Basis

Since $T(\mathbf{x}) = P\mathbf{x}$ then $T(\mathbf{e}_i) = \mathbf{v}_i$, ie, T maps standard basis vector to new basis vectors

Example

Rotate basis in $\mathbb{R}^2$ by $\pi/4$ anticlockwise, find coordinates of a vector wrt the new basis.

$$A_T = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since the matrix A_T rotates $\{\mathbf{e}_1, \mathbf{e}_2\}$, then $A_T = P$ and its columns tell us the coordinates of the new basis and $\mathbf{v} = P[\mathbf{v}]_B$ and $[\mathbf{v}]_B = P^{-1}\mathbf{v}$. The inverse is a rotation clockwise:

$$P^{-1} = \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Example (cntd)

Find the new coordinates of a vector $\mathbf{x} = [1, 1]^T$

$$[\mathbf{x}]_B = P^{-1}\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Change of basis from B to B'

Given an old basis B of $\mathbb{R}^n$ with transition matrix P_B,
and a new basis B' with transition matrix $P_{B'}$,
how do we change from coords in the basis B to coords in the basis B'?

coordinates in B $\xrightarrow{\mathbf{v}=P_B[\mathbf{v}]_B}$ standard coordinates $\xrightarrow{[\mathbf{v}]_{B'}=P_{B'}^{-1}\mathbf{v}}$ coordinates in B'

$$[\mathbf{v}]_{B'} = P_{B'}^{-1} P_B [\mathbf{v}]_B$$

$$M = P_{B'}^{-1} P_B = P_{B'}^{-1}[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n] = [P_{B'}^{-1}\mathbf{v}_1 \quad P_{B'}^{-1}\mathbf{v}_2 \quad \dots \quad P_{B'}^{-1}\mathbf{v}_n]$$

i.e., the columns of the transition matrix M from the old basis B to the new basis B' are the coordinate vectors of the old basis B with respect to the new basis B'

Change of basis from B to B'

Theorem

If B and B' are two bases of $\mathbb{R}^n$, with

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

then the transition matrix from B coordinates to B' coordinates is given by

$$M = \begin{bmatrix} [\mathbf{v}_1]_{B'} & [\mathbf{v}_2]_{B'} & \cdots & [\mathbf{v}_n]_{B'} \end{bmatrix}$$

(i.e., the columns of the transition matrix M from the old basis B to the new basis B' are the coordinate vectors of the old basis B with respect to the new basis B')

Example

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \qquad B' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

are basis of $\mathbb{R}^2$, indeed the corresponding transition matrices from standard basis:

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

have $\det(P) = 3$, $\det(Q) = 1$. Hence, lin. indep. vectors.
We are given

$$[\mathbf{x}]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}_B$$

find its coordinates in B'.

Example (cntd)

1. find first the standard coordinates of $\mathbf{x}$

$$\mathbf{x} = 4\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$$

   and then find B' coordinates:

$$[\mathbf{x}]_{B'} = Q^{-1}\mathbf{x} = \begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix} = \begin{bmatrix}-25\\16\end{bmatrix}_{B'}$$

2. use transition matrix M from B to B' coordinates:
   $\mathbf{v} = P[\mathbf{v}]_B$ and $\mathbf{v} = Q[\mathbf{v}]_{B'}$ $\rightsquigarrow$ $[\mathbf{v}]_{B'} = Q^{-1}P[\mathbf{v}]_B$:

$$M = Q^{-1}P = \begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix} = \begin{bmatrix}-8 & -7\\5 & 4\end{bmatrix}$$

$$[\mathbf{x}]_{B'} = \begin{bmatrix}-8 & -7\\5 & 4\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix} = \begin{bmatrix}-25\\16\end{bmatrix}_{B'}$$