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Linear and Integer Programming

Lecture 10
IP Modeling
Formulations, Relaxations

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Outline

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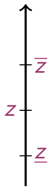
Optimality and Relaxation

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$



How can we prove that $\mathbf{x}^*$ is optimal?
$\overline{z}$ is UB
$\underline{z}$ is LB
stop when $\overline{z} - \underline{z} \leq \epsilon$

- Primal bounds (here lower bounds): every feasible solution gives a primal bound may be easy or hard to find, heuristics
- Dual bounds (here upper bounds): Relaxations

Optimality gap (SCIP):

- If primal and dual bound have opposite signs, the gap is "Infinity".
- If primal and dual bound have the same sign, the gap is

$$\frac{|pb - db|}{\min(|pb|, |db|)}$$

decreases monotonously during the solving process.

Proposition

$$(RP)\ z^R = \max\{f(\mathbf{x}) : \mathbf{x} \in T \subseteq \mathbb{R}^n\} \textit{ is a relaxation of}$$
$$(IP)\ z = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{R}^n\} \textit{ if :}$$

(i) $X \subseteq T$ *or*

(ii) $f(\mathbf{x}) \geq c(\mathbf{x})\ \forall \mathbf{x} \in X$

In other terms:

$$\max_{\mathbf{x}\in T} f(\mathbf{x}) \geq \left\{ \begin{array}{l} \max_{\mathbf{x}\in T} c(\mathbf{x}) \\ \max_{\mathbf{x}\in X} f(\mathbf{x}) \end{array} \right\} \geq \max_{\mathbf{x}\in X} c(\mathbf{x})$$

- T: candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(\mathbf{x}) \geq c(\mathbf{x})$

Relaxations

How to construct relaxations?

1. $IP : \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P \cap \mathbb{Z}^n\}, P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$
   $LP : \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P\}$
   Better formulations give better bounds ($P_1 \subseteq P_2$)

   Proposition

   (i) *If a relaxation RP is infeasible, the original problem IP is infeasible.*

   (ii) *Let $\mathbf{x}^*$ be optimal solution for RP. If $\mathbf{x}^* \in X$ and $f(\mathbf{x}^*) = c(\mathbf{x}^*)$ then $\mathbf{x}^*$ is optimal for IP.*

2. Combinatorial relaxations to easy problems that can be solved rapidly
   Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$IP: \quad z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

$$LR: \quad z(\mathbf{u}) = \max\{\mathbf{c}^T\mathbf{x} + \mathbf{u}(\mathbf{b} - A\mathbf{x}) : \mathbf{x} \in X\}$$

$$z(\mathbf{u}) \geq z \qquad \forall \mathbf{u} \geq \mathbf{0}$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X\} \qquad w = \min\{w(\mathbf{u}) : \mathbf{u} \in U\}$$

form a weak-dual pair if $c(\mathbf{x}) \leq w(\mathbf{u})$ for all $\mathbf{x} \in X$ and all $\mathbf{u} \in U$.
When $z = w$ they form a strong-dual pair

Proposition

$z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_+\}$ *and* $w^{LP} = \min\{\mathbf{u}^T\mathbf{b} : A^T\mathbf{u} \geq \mathbf{c}, \mathbf{u} \in \mathbb{R}^m_+\}$
(ie, dual of linear relaxation) form a weak-dual pair.

Proposition

Let IP and D be weak-dual pair:

(i) *If D is unbounded, then IP is infeasible*

(ii) *If* $\mathbf{x}^* \in X$ *and* $\mathbf{u}^* \in U$ *satisfy* $c(\mathbf{x}^*) = w(\mathbf{u}^*)$ *then* $\mathbf{x}^*$ *is optimal for IP and* $\mathbf{u}^*$ *is optimal for D.*

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Examples

Weak pairs:

Matching: $z = \max\{\mathbf{1}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^m\}$

V. Covering: $w = \min\{\mathbf{1}^T\mathbf{y} : A^T y \geq \mathbf{1}, \mathbf{y} \in \mathbb{Z}_+^n\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:

S. Packing: $z = \max\{\mathbf{1}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^n\}$

S. Covering: $w = \min\{\mathbf{1}^T\mathbf{y} : A^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \in \mathbb{Z}_+^m\}$