

DM545/DM871

Linear and Integer Programming

Linear Programming

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Outline

1. Introduction

Diet Problem

2. Solving LP Problems

Fourier-Motzkin method

3. Preliminaries

Fundamental Theorem of LP

Gaussian Elimination

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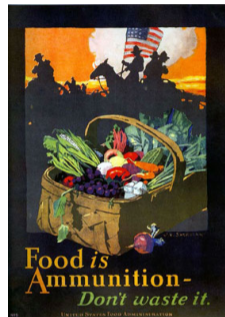
3. Preliminaries

Fundamental Theorem of LP

Gaussian Elimination

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirements at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a **linear programming problem** by George Stigler
- (programming intended as planning not computer code)



min cost/weight

subject to nutrition requirements:

eat enough but not too much of Vitamin A

eat enough but not too much of Sodium

eat enough but not too much of Calories

...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread, and
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5,000 and 50,000)

Food	Cost per serving	Vitamin A	Calories
Corn	\$0.18	107	72
2% Milk	\$0.23	500	121
Wheat Bread	\$0.05	0	65

The Mathematical Model

Parameters (given data)

F = set of foods

N = set of nutrients

a_{ij} = amount of nutrient i in food j , $\forall i \in N, \forall j \in F$

c_j = cost per serving of food j , $\forall j \in F$

$F_{min,j}$ = minimum number of required servings of food j , $\forall j \in F$

$F_{max,j}$ = maximum allowable number of servings of food j , $\forall j \in F$

$N_{min,i}$ = minimum required level of nutrient i , $\forall i \in N$

$N_{max,i}$ = maximum allowable level of nutrient i , $\forall i \in N$

Decision Variables

x_j = number of servings of food j to purchase/consume, $\forall j \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize } \sum_{j \in F} c_j x_j$$

Constraint Set 1: For each nutrient $i \in N$, at least meet the minimum required level

$$\sum_{j \in F} a_{ij} x_j \geq N_{min,i}, \quad \forall i \in N$$

Constraint Set 2: For each nutrient $i \in N$, do not exceed the maximum allowable level.

$$\sum_{j \in F} a_{ij} x_j \leq N_{max,i}, \quad \forall i \in N$$

Constraint Set 3: For each food $j \in F$, select at least the minimum required number of servings

$$x_j \geq F_{min,j}, \quad \forall j \in F$$

Constraint Set 4: For each food $j \in F$, do not exceed the maximum allowable number of servings.

$$x_j \leq F_{max,j}, \quad \forall j \in F$$

The Mathematical Model

system of equalities and inequalities

$$\min \sum_{j \in F} c_j x_j$$

$$\sum_{j \in F} a_{ij} x_j \geq N_{\min, i}, \quad \forall i \in N$$

$$\sum_{j \in F} a_{ij} x_j \leq N_{\max, i}, \quad \forall i \in N$$

$$x_j \geq F_{\min, j}, \quad \forall j \in F$$

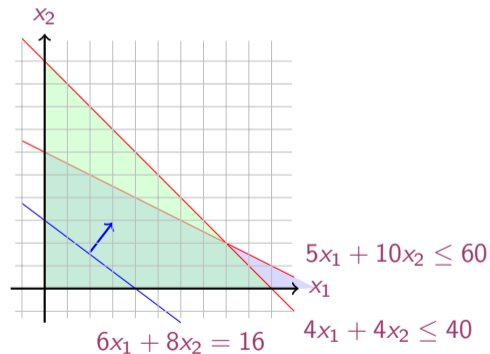
$$x_j \leq F_{\max, j}, \quad \forall j \in F$$

Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Graphical Representation:



In Matrix Form

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{aligned} \max \quad & z = \mathbf{c}^T \mathbf{x} \\ & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Linear Programming

Abstract mathematical model:
Parameters, Decision Variables, Objective, Constraints
(+ Domains & Quantifiers)

The Syntax of a Linear Programming Problem

objective func.	$\max / \min \mathbf{c}^T \mathbf{x}$	$\mathbf{c} \in \mathbb{R}^n$
constraints	s.t. $A\mathbf{x} \begin{matrix} \geq \\ \leq \\ \geq \\ \leq \end{matrix} \mathbf{b}$	$A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$
	$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \in \mathbb{R}^n, \mathbf{0} \in \mathbb{R}^n$

Essential features: continuity, linearity (proportionality and additivity), certainty of parameters

- Any vector $\mathbf{x} \in \mathbb{R}^n$ satisfying all constraints is a **feasible solution**.
- Each $\mathbf{x}^* \in \mathbb{R}^n$ that gives the best possible value for $\mathbf{c}^T \mathbf{x}$ among all feasible \mathbf{x} is an **optimal solution** or **optimum**
- The value $\mathbf{c}^T \mathbf{x}^*$ is the **optimum value**

Diet Problem — History

- The linear programming model consisted of 9 equations in 77 variables
- Stigler, guessed an optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution
- The original instance: <http://www.gams.com/modlib/libhtml/diet.htm>

AMPL Model

```
# diet.mod
set NUTR;
set FOOD;

param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max { j in FOOD } >= f_min[j];
param n_min { NUTR } >= 0;
param n_max { i in NUTR } >= n_min[i];
param amt {NUTR,FOOD} >= 0;

var Buy { j in FOOD } >= f_min[j], <= f_max[j]

minimize total_cost: sum { j in FOOD } cost [j] * Buy[j];
subject to diet { i in NUTR }:
    n_min[i] <= sum { j in FOOD } amt[i,j] * Buy[j] <= n_max[i];
```

AMPL Model

```
# diet.dat
data;

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH MTL SPG TUR;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
  A 700 10000
  C 700 10000
  B1 700 10000
  B2 700 10000 ;

# %
```

```
param amt (tr):
  A C B1 B2 :=
  BEEF 60 20 10 15
  CHK 8 0 20 20
  FISH 8 10 15 10
  HAM 40 40 35 10
  MCH 15 35 15 15
  MTL 70 30 15 15
  SPG 25 50 25 15
  TUR 60 20 15 10 ;
```

Python Script Model

```
# Model diet.py
m = Model("diet")

# Create decision variables for the foods to buy
buy = {}
for f in foods:
    buy[f] = m.addVar(obj=cost[f], name=f)

# Nutrition constraints
for c in categories:
    m.addConstr(
        quicksum(nutritionValues[f,c] * buy[f] for f in foods) <= maxNutrition[c], name=c+'max')
    m.addConstr(
        quicksum(nutritionValues[f,c] * buy[f] for f in foods) >= minNutrition[c], name=c+'min')

# Solve
m.optimize()
```


Python Script

Data

```
from gurobipy import *

categories, minNutrition, maxNutrition = multidict({
    'calories': [1800, 2200],
    'protein': [91, GRB.INFINITY],
    'fat': [0, 65],
    'sodium': [0, 1779] })

foods, cost = multidict({
    'hamburger': 2.49,
    'chicken': 2.89,
    'hot dog': 1.50,
    'fries': 1.89,
    'macaroni': 2.09,
    'pizza': 1.99,
    'salad': 2.49,
    'milk': 0.89,
    'ice cream': 1.59 })
```

```
# Nutrition values for the foods
nutritionValues = {
    ('hamburger', 'calories'): 410,
    ('hamburger', 'protein'): 24,
    ('hamburger', 'fat'): 26,
    ('hamburger', 'sodium'): 730,
    ('chicken', 'calories'): 420,
    ('chicken', 'protein'): 32,
    ('chicken', 'fat'): 10,
    ('chicken', 'sodium'): 1190,
    ('hot dog', 'calories'): 560,
    ('hot dog', 'protein'): 20,
    ('hot dog', 'fat'): 32,
    ('hot dog', 'sodium'): 1800,
    ('fries', 'calories'): 380,
    ('fries', 'protein'): 4,
    ('fries', 'fat'): 19,
    ('fries', 'sodium'): 270,
    ('macaroni', 'calories'): 320,
    ('macaroni', 'protein'): 12,
    ('macaroni', 'fat'): 10,
    ('macaroni', 'sodium'): 930,
    ('pizza', 'calories'): 320,
    ('pizza', 'protein'): 15,
    ...
```

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History of Linear Programming (LP)

System of linear equations

↪ It is impossible to find out who knew what when first.

Just two "references":

- Egyptians and Babylonians considered about 2000 B.C. the solution of special linear equations. But, of course, they described examples and did not describe the methods in "today's style".
- What we call "[Gaussian elimination](#)" today has been explicitly described in Chinese "Nine Books of Arithmetic" which is a compendium written in the period 210 B.C. to A.D. 9, but the methods were probably known long before that.
- Gauss, by the way, never described "Gaussian elimination". He just used it and stated that the linear equations he used can be solved "per eliminationem vulgarem"

History of Linear Programming (LP)

- Origins date back to Newton, Leibnitz, Lagrange, etc.
- In 1827, Fourier described a variable elimination method for **systems of linear inequalities**, today often called Fourier-Moutzkin elimination (Motzkin, 1937). It can be turned into an LP solver but inefficient.
- In 1932, Leontief (1905-1999) Input-Output model to represent interdependencies between branches of a national economy (1976 Nobel prize)
- In 1939, Kantorovich (1912-1986): Foundations of linear programming (Nobel prize in economics with Koopmans on LP, 1975) on Optimal use of scarce resources: foundation and economic interpretation of LP
- The math subfield of **Linear Programming** was created by George Dantzig, John von Neumann (Princeton), and Leonid Kantorovich in the 1940s.
- In 1947, Dantzig (1914-2005) invented the **(primal) simplex algorithm** working for the US Air Force at the Pentagon. (program=plan)

History of LP (cntd)

- In 1954, Lemke: dual simplex algorithm, In 1954, Dantzig and Orchard Hays: revised simplex algorithm
- In 1970, Victor Klee and George Minty created an example that showed that the classical simplex algorithm has exponential worst-case behavior.
- In 1979, L. Khachain found a new **efficient** algorithm for linear programming. It was terribly slow. (Ellipsoid method)
- In 1984, Karmarkar discovered yet another new **efficient** algorithm for linear programming. It proved to be a strong competitor for the simplex method. (Interior point method)

History of Optimization

- In 1951, Nonlinear Programming began with the Karush-Kuhn-Tucker Conditions
- In 1952, Commercial Applications and Software began
- In 1950s, Network Flow Theory began with the work of Ford and Fulkerson.
- In 1955, Stochastic Programming began
- In 1958, Integer Programming began by R. E. Gomory.
- In 1962, Complementary Pivot Theory

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Fourier Motzkin elimination method

Has $Ax \leq b$ a solution? (Assumption: $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^n$)

Idea:

1. transform the system into another by eliminating some variables such that the two systems have the same solutions over the remaining variables.
2. reduce to a system of constant inequalities that can be easily decided

Let x_r be the variable to eliminate

Let $M = \{1 \dots m\}$ index the constraints

For a variable j let's partition the rows of the matrix in

$$N = \{i \in M \mid a_{ij} < 0\}$$

$$Z = \{i \in M \mid a_{ij} = 0\}$$

$$P = \{i \in M \mid a_{ij} > 0\}$$

$$\left\{ \begin{array}{l} x_r \geq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, \quad a_{ir} < 0 \\ x_r \leq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, \quad a_{ir} > 0 \\ \text{all other constraints} \quad i \in Z \end{array} \right. \quad \left\{ \begin{array}{l} x_r \geq A_i(x_1, \dots, x_{r-1}), \quad i \in N \\ x_r \leq B_i(x_1, \dots, x_{r-1}), \quad i \in P \\ \text{all other constraints} \quad i \in Z \end{array} \right.$$

Hence the original system is equivalent to

$$\left\{ \begin{array}{l} \max\{A_i(x_1, \dots, x_{r-1}), i \in N\} \leq x_r \leq \min\{B_i(x_1, \dots, x_{r-1}), i \in P\} \\ \text{all other constraints} \quad i \in Z \end{array} \right.$$

which is equivalent to

$$\left\{ \begin{array}{l} A_i(x_1, \dots, x_{r-1}) \leq B_j(x_1, \dots, x_{r-1}) \quad i \in N, j \in P \\ \text{all other constraints} \quad i \in Z \end{array} \right.$$

we eliminated x_r but:

$$\left\{ \begin{array}{l} |N| \cdot |P| \text{ inequalities} \\ |Z| \text{ inequalities} \end{array} \right.$$

after d iterations if $|P| = |N| = m/2$ exponential growth: $(1/4^d)(m/2)^{2^d}$

Example

$$-7x_1 + 6x_2 \leq 25$$

$$x_1 - 5x_2 \leq 1$$

$$x_1 \leq 7$$

$$-x_1 + 2x_2 \leq 12$$

$$-x_1 - 3x_2 \leq 1$$

$$2x_1 - x_2 \leq 10$$

x_2 variable to eliminate

$$N = \{2, 5, 6\}, Z = \{3\}, P = \{1, 4\}$$

$$|Z \cup (N \times P)| = 7 \text{ constraints}$$

By adding one variable and one inequality, Fourier-Motzkin elimination can be turned into an LP solver.

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Definitions

- \mathbb{R} : set of real numbers
 $\mathbb{N} = \{1, 2, 3, 4, \dots\}$: set of natural numbers (positive integers)
 $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$: set of all integers
 $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$: set of rational numbers
- column vector and matrices
 scalar product: $\mathbf{y}^T \mathbf{x} = \sum_{i=1}^n y_i x_i$

- linear combination

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n \quad \mathbf{x} = \lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k = \sum_{i=1}^k \lambda_i \mathbf{v}_i$$

$$\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_k]^T \in \mathbb{R}^k$$

moreover:

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

$$\boldsymbol{\lambda}^T \mathbf{1} = 1$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \text{ and } \boldsymbol{\lambda}^T \mathbf{1} = 1$$

conic combination

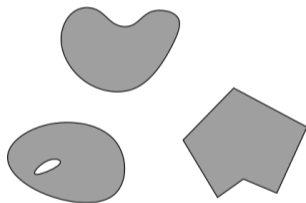
affine combination

convex combination

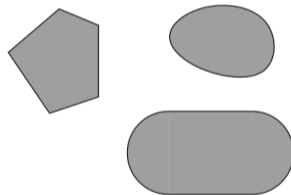
$$\left(\sum_{i=1}^k \lambda_i = 1 \right)$$

Definitions

- set S is **linear (affine) independent** if no element of it can be expressed as linear combination of the others
Eg: $S \subseteq \mathbb{R}^n \implies \max n$ lin. indep. ($\max n + 1$ aff. indep.)
- **convex set**: if $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq \lambda \leq 1$ then $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in S$



nonconvex

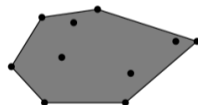


convex

- **convex function** if its epigraph $\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is a convex set or $f : X \rightarrow \mathbb{R}$, or if $\forall x, y \in X, \lambda \in [0, 1]$ it holds that $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$

Definitions

- For a set of points $S \subseteq \mathbb{R}^n$
 - $\text{lin}(S)$ linear hull (span)
 - $\text{cone}(S)$ conic hull
 - $\text{aff}(S)$ affine hull
 - $\text{conv}(S)$ convex hull



the convex hull of X

$$\text{conv}(X) = \{ \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_n \mathbf{x}_n \mid \mathbf{x}_i \in X, \lambda_1, \dots, \lambda_n \geq 0 \text{ and } \sum_i \lambda_i = 1 \}$$

Definitions

- **rank** of a matrix for columns (= for rows)
if (m, n) -matrix has rank = $\min\{m, n\}$ then the matrix is full rank
if (n, n) -matrix is full rank then it is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an **hyperplane** if $\exists \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

$$G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T \mathbf{x} = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an **halfspace** if $\exists \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T \mathbf{x} \leq \alpha\}$$

($\mathbf{a}^T \mathbf{x} = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subset \mathbb{R}^n$ is a **polyhedron** if $\exists m \in \mathbb{Z}^+, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$:

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\} = \bigcap_{i=1}^m \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i \cdot \mathbf{x} \leq b_i\}$$

i.e., a polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

- a polyhedron P is a **polytope** if it is bounded: $\exists B \in \mathbb{R}, B > 0$:

$$P \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq B\}$$

- A set of vectors is a **polytope** if it is the convex hull of finitely many vectors.

Definitions

- General optimization problem: $\max\{\varphi(\mathbf{x}) \mid \mathbf{x} \in F\}$, F is feasible region for \mathbf{x}
- Note: if F is open, eg, $x < 5$ then: $\sup\{x \mid x < 5\}$
supremum: least element of \mathbb{R} greater or equal than any element in F
- If A and \mathbf{b} are made of rational numbers, $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ is a rational polyhedron

Definitions

- The inequality denoted by (\mathbf{a}, α) is called a **valid inequality for P** if $\mathbf{a}\mathbf{x} \leq \alpha, \forall \mathbf{x} \in P$.
Note that (\mathbf{a}, α) is a valid inequality if and only if P lies in the half-space $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}\mathbf{x} \leq \alpha\}$.
- A **face** of P is $F = \{\mathbf{x} \in P \mid \mathbf{a}\mathbf{x} = \alpha\}$ where (\mathbf{a}, α) is a valid inequality for P . Hence, it is the intersection of P with the hyperplane of a valid inequality. It is said to be **proper** if $F \neq \emptyset$ and $F \neq P$.
- If $F \neq \emptyset$ we say that it **supports P** .
If \mathbf{c} is a non zero vector for which $\delta = \max\{\mathbf{c}^T \mathbf{x} \mid \mathbf{x} \in P\}$ is finite, then the set $\{\mathbf{x} \mid \mathbf{c}^T \mathbf{x} = \delta\}$ is called **supporting hyperplane**.
- A point \mathbf{x} for which $\{\mathbf{x}\}$ is a face is called a **vertex** of P and also a **basic solution** of $A\mathbf{x} \leq \mathbf{b}$ (**0 dim** face)
- A **facet** is a maximal face distinct from P
 $\mathbf{c}\mathbf{x} \leq \mathbf{d}$ is facet defining if $\mathbf{c}\mathbf{x} = \mathbf{d}$ is a supporting hyperplane of P of $n - 1$ dim

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$

Task:

1. decide that $\{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} \leq \mathbf{b}\}$ is empty (prob. infeasible), or
2. find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T \mathbf{x}$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T \mathbf{x} > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and \exists solution
 1. one solution
 2. infinite solutions
3. $F \neq \emptyset$ and \nexists solution

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Fundamental Theorem of LP

Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{\mathbf{c}^T \mathbf{x} \mid \mathbf{x} \in P\} \text{ where } P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

If P is a bounded polyhedron and not empty and \mathbf{x}^* is an optimal solution to the problem, then:

- \mathbf{x}^* is an extreme point (vertex) of P , or
- \mathbf{x}^* lies on a face $F \subset P$ of optimal solutions



Proof idea:

- assume \mathbf{x}^* not a vertex of P then \exists a ball around it still in P . Show that a point in the ball has better cost
- if \mathbf{x}^* is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

Implications:

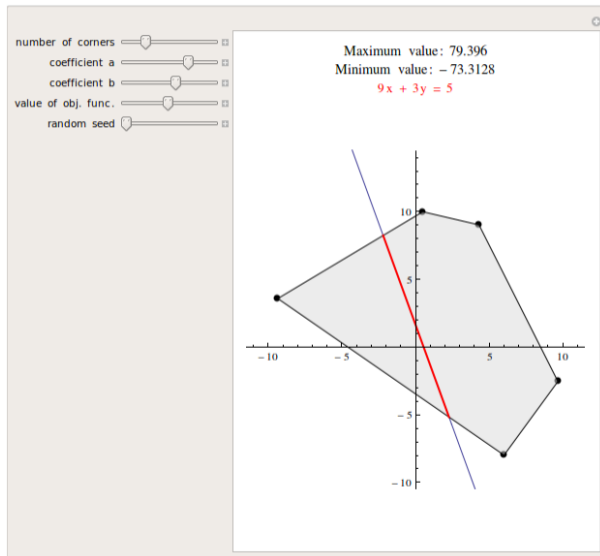
- the optimal solution is at the intersection of supporting hyperplanes.
- hence finitely many possibilities
- solution method: write all inequalities as equalities and solve all $\binom{m}{n}$ systems of linear equalities (n # variables, m # equality constraints)
- for each point we then need to check if feasible and if best in cost.
- each system is solved by Gaussian elimination
- Stirling approximation:

$$\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}} \text{ as } m \rightarrow \infty$$

Simplex Method

1. find a solution that is at the intersection of some n hyperplanes
2. try systematically to produce the other points by exchanging one hyperplane with another
3. check optimality, proof provided by duality theory

Demo



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Gaussian Elimination

Gaussian Elimination

1. Forward elimination

reduces the system to row echelon form by elementary row operations

- multiply a row by a non-zero constant
- interchange two rows
- add a multiple of one row to another

(or LU decomposition)

2. Back substitution (or reduced row echelon form - RREF)

Example

$$\begin{aligned}2x + y - z &= 8 & (R1) \\ -3x - y + 2z &= -11 & (R2) \\ -2x + y + 2z &= -3 & (R3)\end{aligned}$$

$$\begin{array}{|c|c|c|c|c|} \hline R1 & 2 & 1 & -1 & 8 \\ \hline R2 & -3 & -1 & 2 & -11 \\ \hline R3 & -2 & 1 & 2 & -3 \\ \hline \end{array}$$

$$\begin{aligned}2x + y - z &= 8 & (R1) \\ + \frac{1}{2}y + \frac{1}{2}z &= 1 & (R2) \\ + 2y + 1z &= 5 & (R3)\end{aligned}$$

$$\begin{array}{|c|c|c|c|c|} \hline R1' = 1/2 R1 & 1 & 1/2 & -1/2 & 4 \\ \hline R2' = R2 + 3/2 R1 & 0 & 1/2 & 1/2 & 1 \\ \hline R3' = R3 + R1 & 0 & 2 & 1 & 5 \\ \hline \end{array}$$

$$\begin{aligned}2x + y - z &= 8 & (R1) \\ + \frac{1}{2}y + \frac{1}{2}z &= 1 & (R2) \\ - z &= 1 & (R3)\end{aligned}$$

$$\begin{array}{|c|c|c|c|c|} \hline R1' = R1 & 1 & 1/2 & -1/2 & 4 \\ \hline R2' = 2 R2 & 0 & 1 & 1 & 2 \\ \hline R3' = R3 - 4 R2 & 0 & 0 & -1 & 1 \\ \hline \end{array}$$

$$\begin{aligned}2x + y - z &= 8 & (R1) \\ + \frac{1}{2}y + \frac{1}{2}z &= 1 & (R2) \\ - z &= 1 & (R3)\end{aligned}$$

$$\begin{array}{|c|c|c|c|c|} \hline R1' = R1 - 1/2 R3 & 1 & 1/2 & 0 & 7/2 \\ \hline R2' = R2 + R3 & 0 & 1 & 0 & 3 \\ \hline R3' = -R3 & 0 & 0 & 1 & -1 \\ \hline \end{array}$$

$$\begin{aligned}x &= 2 & (R1) \\ y &= 3 & (R2) \\ z &= -1 & (R3)\end{aligned}$$

$$\begin{array}{|c|c|c|c|c|} \hline R1' = R1 - 1/2 R2 & 1 & 0 & 0 & 2 & \Rightarrow x=2 \\ \hline R2' = R2 & 0 & 1 & 0 & 3 & \Rightarrow y=3 \\ \hline R3' = R3 & 0 & 0 & 1 & -1 & \Rightarrow z=-1 \\ \hline \end{array}$$

LU Factorization

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$Ax = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = PLU$$

$$\mathbf{x} = A^{-1}\mathbf{b} = U^{-1}L^{-1}P^T\mathbf{b}$$

$$\mathbf{z}_1 = P^T\mathbf{b}, \quad \mathbf{z}_2 = L^{-1}\mathbf{z}_1, \quad \mathbf{x} = U^{-1}\mathbf{z}_2$$

```
In [36]: import pprint
...: import scipy
...:
...: A = scipy.array([ [2, 1, -1], [-3, -1, 2], [-2, 1, 2] ])
...: P,L,U = scipy.linalg.lu(A)
...:
...: print(P)
...: print(L)
...: print(U)
```

```
[[0. 0. 1.]
 [1. 0. 0.]
 [0. 1. 0.]]
[[ 1.          0.          0.          ]
 [ 0.66666667  1.          0.          ]
 [-0.66666667  0.2         1.          ]]
[[ -3.         -1.          2.          ]
 [ 0.          1.66666667  0.66666667 ]
 [ 0.          0.          0.2         ]]
```

Polynomial time $O(n^2m)$ but needs to guarantee that all the numbers during the run can be represented by polynomially bounded bits

Summary

1. Introduction

Diet Problem

2. Solving LP Problems

Fourier-Motzkin method

3. Preliminaries

Fundamental Theorem of LP

Gaussian Elimination