DM545/DM871

Linear and Integer Programming

## Lecture 6

# More on Duality

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# Outline

# Summary

- Derivation:
  1. economic interpretation
  2. bounding
  3. multipliers
  4. recipe
  5. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
- Dual Simplex
- Sensitivity Analysis, Economic interpretation

# Outline

1. Derivation
   - Lagrangian Duality
2. Dual Simplex
3. Sensitivity Analysis

# Outline

# Lagrangian Duality

Relaxation: if a problem is hard to solve then find an easier problem resembling the original one that provides information in terms of bounds. Then, search for the strongest bounds.

$$
\begin{array}{rcrcrcrcl}
\min 13x_1 & + & 6x_2 & + & 4x_3 & + & 12x_4 & & \\
2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & = & 7 \\
3x_1 & + & & + & 2x_3 & + & 4x_4 & = & 2 \\
& & & & & x_1, & x_2, x_3, x_4 & \geq & 0
\end{array}
$$

We wish to reduce to a problem easier to solve, ie:

$$
\begin{array}{r}
\min c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
x_1, x_2, \ldots, x_n \geq 0
\end{array}
$$

solvable by inspection: if $c < 0$ then $x = +\infty$, if $c \geq 0$ then $x = 0$.
measure of violation of the constraints:

$$
\begin{array}{l}
7 - (2x_1 + 3x_2 + 4x_3 + 5x_4) \\
2 - (3x_1 + \qquad + 2x_3 + 4x_4)
\end{array}
$$

We relax these measures in obj. function with Lagrangian multipliers $y_1$, $y_2$.
We obtain a family of problems:

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{rrrrr} & 13x_1 & + 6x_2 & + 4x_3 & + 12x_4 \\ +y_1(7- & 2x_1 & - 3x_2 & - 4x_3 & - 5x_4) \\ +y_2(2- & 3x_1 & & - 2x_3 & - 4x_4) \end{array} \right\}$$

1. for all $y_1, y_2 \in \mathbb{R} : \mathrm{opt}(PR(y_1, y_2)) \leq \mathrm{opt}(P)$
2. $\max_{y_1, y_2 \in \mathbb{R}} \{\mathrm{opt}(PR(y_1, y_2))\} \leq \mathrm{opt}(P)$

PR is easy to solve.
(It can be also seen as a proof of the weak duality theorem)

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{rl} & (13 - 2y_2 - 3y_2)\ x_1 \\ + & (6 - 3y_1 \qquad\qquad)\ x_2 \\ + & (4 \qquad\qquad - 2y_2)\ x_3 \\ + & (12 - 5y_1 - 4y_2)\ x_4 \\ + & \qquad 7y_1 + 2y_2 \end{array} \right\}$$

if coeff. of $x$ is $< 0$ then bound is $-\infty$ then LB is useless

$$\begin{array}{r} (13 - 2y_2 - 3y_2) \geq 0 \\ (6 - 3y_1 \qquad\qquad) \geq 0 \\ (4 \qquad\qquad - 2y_2) \geq 0 \\ (12 - 5y_1 - 4y_2) \geq 0 \end{array}$$

If they all hold then we are left with $7y_1 + 2y_2$ because all go to 0.

$$\begin{array}{rrrl} \max & 7y_1 & + 2y_2 & \\ & 2y_2 & + 3y_2 & \leq 13 \\ & 3y_1 & & \leq 6 \\ & & + 2y_2 & \leq 4 \\ & 5y_1 & + 4y_2 & \leq 12 \end{array}$$

# General Formulation

$$\begin{aligned} \min \quad & z = \mathbf{c}^T\mathbf{x} & \qquad & \mathbf{c} \in \mathbb{R}^n \\ & A\mathbf{x} = \mathbf{b} & & A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m \\ & \mathbf{x} \geq \mathbf{0} & & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

$$\max_{\mathbf{y}\in\mathbb{R}^m}\{\min_{\mathbf{x}\in\mathbb{R}^n_+}\{\mathbf{c}^T\mathbf{x} + \mathbf{y}^T(\mathbf{b} - A\mathbf{x})\}\}$$

$$\max_{\mathbf{y}\in\mathbb{R}^m}\{\min_{\mathbf{x}\in\mathbb{R}^n_+}\{(\mathbf{c}^T - \mathbf{y}^T A)\mathbf{x} + \mathbf{y}^T\mathbf{b}\}\}$$

$$\begin{aligned} \max \quad & \mathbf{b}^T\mathbf{y} \\ & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \in \mathbb{R}^m \end{aligned}$$

# Outline

# Dual Simplex

- Dual simplex (Lemke, 1954): apply the simplex method to the dual problem and observe what happens in the primal tableau:

$$\begin{aligned}\max\{c^T x \mid Ax \le b, x \ge 0\} &= \min\{b^T y \mid A^T y \ge c^T, y \ge 0\} \\ &= -\max\{-b^T y \mid -A^T x \le -c^T, y \ge 0\}\end{aligned}$$

- We obtain a new algorithm for the primal problem: the dual simplex
  It corresponds to the primal simplex applied to the dual

Primal simplex on primal problem:

1. pivot $> 0$
2. col $c_j$ with wrong sign
3. row: $\min\left\{\frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, .., m\right\}$

Dual simplex on primal problem:

1. pivot $< 0$
2. row $b_i < 0$
   (condition of feasibility)
3. col: $\min\left\{\left|\frac{c_j}{a_{ij}}\right| : a_{ij} < 0, j = 1, 2, .., n + m\right\}$
   (least worsening solution)

# Dual Simplex



1. (primal) simplex on primal problem (the one studied so far)

2. Now: dual simplex on primal problem $\equiv$ primal simplex on dual problem
   (implemented as dual simplex, understood as primal simplex on dual problem)

Uses of 1.:

- The dual simplex can work better than the primal in some cases.
  Eg. since running time in practice between $2m$ and $3m$, then if $m = 99$ and $n = 9$ then better the dual

- Infeasible start
  Dual based Phase I algorithm (Dual-primal algorithm)

# Dual Simplex for Phase I

## Example

Primal:

$$\begin{array}{rrcrcr} \max & -x_1 & - & x_2 & & \\ & -2x_1 & - & x_2 & \le & 4 \\ & -2x_1 & + & 4x_2 & \le & -8 \\ & -x_1 & + & 3x_2 & \le & -7 \\ & & & x_1, x_2 & \ge & 0 \end{array}$$

- Initial tableau

```
|   | x1 | x2 | w1 | w2 | w3 | -z |  b |
|---+----+----+----+----+----+----+----|
|   | -2 | -1 |  1 |  0 |  0 |  0 |  4 |
|   | -2 |  4 |  0 |  1 |  0 |  0 | -8 |
|   | -1 |  3 |  0 |  0 |  1 |  0 | -7 |
|---+----+----+----+----+----+----+----|
|   | -1 | -1 |  0 |  0 |  0 |  1 |  0 |
```

  infeasible start

- $x_1$ enters, $w_2$ leaves

Dual:

$$\begin{array}{rrcrcrcr} \min & 4y_1 & - & 8y_2 & - & 7y_3 & & \\ & -2y_1 & - & 2y_2 & - & y_3 & \ge & -1 \\ & -y_1 & + & 4y_2 & + & 3y_3 & \ge & -1 \\ & & & & y_1, y_2, & y_3 & \ge & 0 \end{array}$$

- Initial tableau ($\min by \equiv -\max -by$)

```
|   | y1 | y2 | y3 | z1 | z2 | -p | b |
|---+----+----+----+----+----+----+---|
|   |  2 |  2 |  1 |  1 |  0 |  0 | 1 |
|   |  1 | -4 | -3 |  0 |  1 |  0 | 1 |
|---+----+----+----+----+----+----+---|
|   | -4 |  8 |  7 |  0 |  0 |  1 | 0 |
```

  feasible start (thanks to $-x_1 - x_2$)

- $y_2$ enters, $z_1$ leaves

- $x_1$ enters, $w_2$ leaves

| | x1 | x2 | w1 | w2 | w3 | -z | b |
|---|---|---|---|---|---|---|---|
| | 0 | -5 | 1 | -1 | 0 | 0 | 12 |
| | 1 | -2 | 0 | -0.5 | 0 | 0 | 4 |
| | 0 | 1 | 0 | -0.5 | 1 | 0 | -3 |
| | 0 | -3 | 0 | -0.5 | 0 | 1 | 4 |

- $w_2$ enters, $w_3$ leaves (note that we kept $c_j < 0$, ie, optimality)

| | x1 | x2 | w1 | w2 | w3 | -z | b |
|---|---|---|---|---|---|---|---|
| | 0 | -7 | 1 | 0 | -2 | 0 | 18 |
| | 1 | -3 | 0 | 0 | -1 | 0 | 7 |
| | 0 | -2 | 0 | 1 | -2 | 0 | 6 |
| | 0 | -4 | 0 | 0 | -1 | 1 | 7 |

- $y_2$ enters, $z_1$ leaves

| | y1 | y2 | y3 | z1 | z2 | -p | b |
|---|---|---|---|---|---|---|---|
| | 1 | 1 | 0.5 | 0.5 | 0 | 0 | 0.5 |
| | 5 | 0 | -1 | 2 | 1 | 0 | 3 |
| | -4 | 0 | 3 | -12 | 0 | 1 | -4 |

- $y_3$ enters, $y_2$ leaves

| | y1 | y2 | y3 | z1 | z2 | -p | b |
|---|---|---|---|---|---|---|---|
| | 2 | 2 | 1 | 1 | 0 | 0 | 1 |
| | 7 | 2 | 0 | 3 | 1 | 0 | 3 |
| | -18 | -6 | 0 | -7 | 0 | 1 | -7 |

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# Outline

# Economic Interpretation

$$
\begin{aligned}
\max\ 5x_0 &+ 6x_1 + 8x_2 \\
6x_0 &+ 5x_1 + 10x_2 \leq 60 \\
8x_0 &+ 4x_1 + 4x_2 \leq 40 \\
4x_0 &+ 5x_1 + 6x_2 \leq 50 \\
& x_0, x_1, x_2 \geq 0
\end{aligned}
$$

final tableau:

| $x0$ | $x1$ | $x2$ | $s1$ | $s2$ | $s3$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | | 0 | | | 5/2 |
| | 1 | 0 | | 0 | | | 7 |
| | 0 | 0 | | 1 | | | 2 |
| $-1/5$ | 0 | 0 | $-1/5$ | 0 | $-1$ | | $-62$ |

- Which are the values of variables, the reduced costs, the shadow prices (or marginal prices), the values of dual variables?
- If one slack variable $> 0$ then overcapacity: $s_2 = 2$ then the second constraint is not tight
- How many products can be produced at most? at most $m$
- How much more expensive a product not selected should be?
  look at reduced costs: $c_j + \boldsymbol{\pi}\mathbf{a}_j > 0$
- What is the value of extra capacity of manpower? In +1 out +1/5

# Economic Interpretation

Game: Suppose two economic operators:

- P owns the factory and produces goods
- D is in the market buying and selling raw material and resources
- D asks P to close and sell him all resources
- P considers if the offer is convenient
- D wants to spend least possible
- $y$ are prices that D offers for the resources
- $\sum y_i b_i$ is the amount D has to pay to have all resources of P
- $\sum y_i a_{ij} \geq c_j$ total value to make $j$ > price per unit of product
- P either sells all resources $\sum y_i a_{ij}$ or produces product $j$ ($c_j$)
- without $\geq$ there would not be negotiation because P would be better off producing and selling

- at optimality the situation is indifferent (strong th.)
- resource 2 that was not totally utilized in the primal has been given value 0 in the dual. (complementary slackness th.) Plausible, since we do not use all the resource, likely to place not so much value on it.
- for product 0 $\sum y_i a_{ij} > c_j$ hence not profitable producing it. (complementary slackness th.)

# Sensitivity Analysis
## aka Postoptimality Analysis

Instead of solving each modified problem from scratch, exploit results obtained from solving the original problem.

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\} \tag{*}$$

(I) changes to coefficients of objective function: $\max\{\tilde{\mathbf{c}}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$ (primal)
$\mathbf{x}^*$ of (*) remains feasible hence we can restart the simplex from $\mathbf{x}^*$

(II) changes to RHS terms: $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \tilde{\mathbf{b}}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$ (dual)
$\mathbf{x}^*$ optimal feasible solution of (*)
basic sol $\bar{\mathbf{x}}$ of (II): $\bar{\mathbf{x}}_N = \mathbf{x}^*_N$, $A_B\bar{\mathbf{x}}_B = \tilde{\mathbf{b}} - A_N\bar{\mathbf{x}}_N$
$\bar{\mathbf{x}}$ is dual feasible and we can start the dual simplex from there. If $\tilde{\mathbf{b}}$ differs from $\mathbf{b}$ only slightly it may be we are already optimal.

(III) introduce a new variable: (primal)

$$\max \sum_{j=1}^{6} c_j x_j$$
$$\sum_{j=1}^{6} a_{ij} x_j = b_i,\ i = 1, \ldots, 3$$
$$l_j \leq x_j \leq u_j,\ j = 1, \ldots, 6$$
$$[x_1^*, \ldots, x_6^*] \text{ feasible}$$

$$\max \sum_{j=1}^{7} c_j x_j$$
$$\sum_{j=1}^{7} a_{ij} x_j = b_i,\ i = 1, \ldots, 3$$
$$l_j \leq x_j \leq u_j,\ j = 1, \ldots, 7$$
$$[x_1^*, \ldots, x_6^*, 0] \text{ feasible}$$

(IV) introduce a new constraint: (dual)

$$\sum_{j=1}^{6} a_{4j} x_j = b_4$$
$$\sum_{j=1}^{6} a_{5j} x_j = b_5$$
$$l_j \leq x_j \leq u_j \qquad j = 7, 8$$

$$[x_1^*, \ldots, x_6^*] \text{ optimal}$$
$$[x_1^*, \ldots, x_6^*, x_7^*, x_8^*] \text{ feasible}$$
$$x_7^* = b_4 - \sum_{j=1}^{6} a_{4j} x_j^*$$
$$x_8^* = b_5 - \sum_{j=1}^{6} a_{5j} x_j^*$$

# Examples

(I) Variation of reduced costs:

$$\begin{aligned} \max\ 6x_1 &+ \ 8x_2 \\ 5x_1 &+ 10x_2 \leq 60 \\ 4x_1 &+ \ 4x_2 \ \leq 40 \\ &x_1, x_2 \geq \ 0 \end{aligned}$$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|
| $x_3$ | 5 | 10 | 1 | 0 | 0 | 60 |
| $x_4$ | 4 | 4 | 0 | 1 | 0 | 40 |
| | 6 | 8 | 0 | 0 | 1 | 0 |

The last tableau gives the possibility to estimate the effect of variations

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | $1/5$ | $-1/4$ | 0 | 2 |
| $x_1$ | 1 | 0 | $-1/5$ | $1/2$ | 0 | 8 |
| | 0 | 0 | $-2/5$ | $-1$ | 1 | $-64$ |

For a variable in basis the perturbation goes unchanged in the red. costs. Eg:

$$\max\ (6+\delta)x_1 + 8x_2 \implies \bar{c}_1 = 1(6+\delta) - \frac{2}{5}\cdot 5 - 1\cdot 4 = \delta$$

then need to bring in canonical form and hence $\delta$ changes the obj value.
For a variable not in basis, if it changes the sign of the reduced cost $\implies$ worth bringing in basis $\implies$the $\delta$ term propagates to other columns

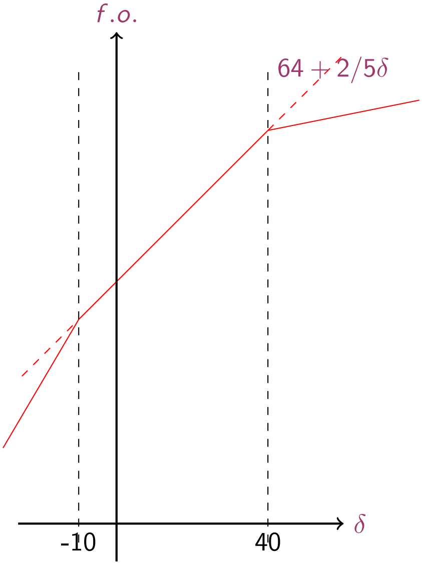
## (II) Changes in RHS terms

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|
| $x_3$ | 5 | 10 | 1 | 0 | 0 | $60+\delta$ |
| $x_4$ | 4 | 4 | 0 | 1 | 0 | $40+\epsilon$ |
| | 6 | 8 | 0 | 0 | 1 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | $1/5$ | $-1/4$ | 0 | $2+1/5\delta-1/4\epsilon$ |
| $x_1$ | 1 | 0 | $-1/5$ | $1/2$ | 0 | $8-1/5\delta+1/2\epsilon$ |
| | 0 | 0 | $-2/5$ | $-1$ | 1 | $-64-2/5\delta-\epsilon$ |

(It would be more convenient to augment the second. But let's take $\epsilon = 0$.)
If $60+\delta \Longrightarrow$all RHS terms change and we must check feasibility
Which are the multipliers for the first row?$k_1 = \frac{1}{5}$, $k_2 = -\frac{1}{4}$, $k_3 = 0$
I: $1/5(60+\delta) - 1/4 \cdot 40 + 0 \cdot 0 = 12 + \delta/5 - 10 = 2 + \delta/5$
II: $-1/5(60+\delta) + 1/2 \cdot 40 + 0 \cdot 0 = -60/5 + 20 - \delta/5 = 8 - 1/5\delta$
Risk that RHS becomes negative
Eg: if $\delta = -10 \Longrightarrow$tableau stays optimal but not feasible $\Longrightarrow$apply dual simplex

# Graphical Representation



(III) Add a variable

$$\begin{aligned} \max\ 5x_0 + 6x_1 + \ 8x_2 & \\ 6x_0 + 5x_1 + 10x_2 &\leq 60 \\ 8x_0 + 4x_1 + \ 4x_2 &\leq 40 \\ x_0, x_1, x_2 &\geq \ 0 \end{aligned}$$

Reduced cost of $x_0$? $c_j + \sum \pi_i a_{ij} = +1 \cdot 5 - \frac{2}{5} \cdot 6 + (-1)8 = -\frac{27}{5}$

To make worth entering in basis:

- increase its cost
- decrease the technological coefficient in constraint II: $5 - 2/5 \cdot 6 - a_{20} > 0$

## (IV) Add a constraint

$$
\begin{aligned}
\max\ 6x_1 &+ \ 8x_2 \\
5x_1 &+ 10x_2 \leq 60 \\
4x_1 &+ \ 4x_2 \ \leq 40 \\
5x_1 &+ \ 6x_2 \ \leq 50 \\
x_1, x_2 &\geq \ 0
\end{aligned}
$$

Final tableau not in canonical form, need to iterate with dual simplex

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | $1/5$ | $-1/4$ | | 0 | 2 |
| $x_1$ | 1 | 0 | $-1/5$ | $1/2$ | | 0 | 8 |
| | 0 | 0 | $5/5$ | $6/4$ | 1 | 0 | $-2$ |
| | 0 | 0 | $-2/5$ | $-1$ | 0 | 1 | $-64$ |

(V) change in a technological coefficient:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|
| $x_3$ | 5 | $10+\delta$ | 1 | 0 | 0 | 60 |
| $x_4$ | 4 | 4 | 0 | 1 | 0 | 40 |
| | 6 | 8 | 0 | 0 | 1 | 0 |

- first effect on its column
- then look at $c$
- finally look at $b$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $-z$ | $b$ |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | $(10+\delta)1/5+4(-1/4)$ | $1/5$ | $-1/4$ | 0 | 2 |
| $x_1$ | 1 | $(10+\delta)(-1/5)+4(1/2)$ | $-1/5$ | $1/2$ | 0 | 8 |
| | 0 | $-2/5\delta$ | $-2/5$ | $-1$ | 1 | $-64$ |

# Relevance of Sensitivity Analysis

- The dominant application of LP is mixed integer linear programming.
- In this context it is extremely important being able to begin with a model instantiated in one form followed by a sequence of problem modifications
  - row and column additions and deletions,
  - variable fixings

  interspersed with resolves

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