

DM545/DM871

Linear and Integer Programming

Lecture 7

## Revised Simplex Method

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# Outline

1. Revised Simplex Method
2. Efficiency Issues

# Motivation

Complexity of single pivot operation in standard simplex:

- entering variable  $O(n)$
- leaving variable  $O(m)$
- updating the tableau  $O(mn)$

Problems with this:

- Time: we are doing operations that are not actually needed  
Space: we need to store the whole tableau:  $O(mn)$  floating point numbers
- Most problems have sparse matrices (many zeros)  
sparse matrices are typically handled efficiently  
the standard simplex has the 'Fill in' effect: sparse matrices are lost
- accumulation of Floating Point Errors over the iterations

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# Revised Simplex Method

Several ways to improve wrt pitfalls in the previous slide, requires matrix description of the simplex.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1..m \\ & x_j \geq 0 \quad j = 1..n \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{A} \in \mathbb{R}^{m \times (n+m)} \\ & \mathbf{c} \in \mathbb{R}^{(n+m)}, \mathbf{b} \in \mathbb{R}^m, \mathbf{x} \in \mathbb{R}^{n+m} \end{aligned}$$

$$\max\{\mathbf{c}^T \mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

At each iteration the simplex moves from a basic feasible solution to another.

For each basic feasible solution:

- $B = \{1 \dots m\}$  basis
- $N = \{m + 1 \dots m + n\}$
- $A_B = [\mathbf{a}_1 \dots \mathbf{a}_m]$  basis matrix
- $A_N = [\mathbf{a}_{m+1} \dots \mathbf{a}_{m+n}]$
- $\mathbf{x}_N = 0$
- $\mathbf{x}_B \geq 0$

$$\left[ \begin{array}{cc|c|c} A_N & A_B & \mathbf{0} & \mathbf{b} \\ \hline \mathbf{c}_N^T & \mathbf{c}_B^T & 1 & 0 \end{array} \right]$$

$$A\mathbf{x} = A_N\mathbf{x}_N + A_B\mathbf{x}_B = \mathbf{b}$$

$$A_B\mathbf{x}_B = \mathbf{b} - A_N\mathbf{x}_N$$

Basic feasible solution  $\iff A_B$  is non-singular

$$\mathbf{x}_B = A_B^{-1}\mathbf{b} - A_B^{-1}A_N\mathbf{x}_N$$

for the objective function:

$$z = \mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N$$

Substituting for  $\mathbf{x}_B$  from above:

$$\begin{aligned} z &= \mathbf{c}_B^T (A_B^{-1} \mathbf{b} - A_B^{-1} A_N \mathbf{x}_N) + \mathbf{c}_N^T \mathbf{x}_N = \\ &= \mathbf{c}_B^T A_B^{-1} \mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N) \mathbf{x}_N \end{aligned}$$

Collecting together:

$$\begin{aligned} \mathbf{x}_B &= A_B^{-1} \mathbf{b} - A_B^{-1} A_N \mathbf{x}_N \\ z &= \mathbf{c}_B^T A_B^{-1} \mathbf{b} + (\mathbf{c}_N^T - \underbrace{\mathbf{c}_B^T A_B^{-1} A_N}_{\bar{A}}) \mathbf{x}_N \end{aligned}$$

In tableau form, for a basic feasible solution corresponding to  $B$  we have:

$$\left[ \begin{array}{ccc|c} & & & \\ & & & \\ & & & \\ & & & \\ \hline \mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N & 0 & 1 & -\mathbf{c}_B^T A_B^{-1} \mathbf{b} \end{array} \right]$$

We do not need to compute all elements of  $\bar{A}$

## Example

$$\begin{aligned} \max \quad & x_1 + x_2 \\ -x_1 + x_2 & \leq 1 \\ x_1 & \leq 3 \\ x_2 & \leq 2 \\ x_1, x_2 & \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ -x_1 + x_2 + x_3 & = 1 \\ x_1 + x_4 & = 3 \\ x_2 + x_5 & = 2 \\ x_1, x_2, x_3, x_4, x_5 & \geq 0 \end{aligned}$$

Initial tableau

x1	x2	x3	x4	x5	-z	b
-1	1	1	0	0	0	1
1	0	0	1	0	0	3
0	1	0	0	1	0	2
1	1	0	0	0	1	0

After two iterations

x1	x2	x3	x4	x5	-z	b
1	0	-1	0	1	0	1
0	1	0	0	1	0	2
0	0	1	1	-1	0	2
0	0	1	0	-2	1	3

Basic variables  $x_1, x_2, x_4$ . Non basic:  $x_3, x_5$ . From the [initial tableau](#):

$$A_B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} \quad x_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$$

$$c_B^T = [1 \ 1 \ 0] \quad c_N^T = [0 \ 0]$$



- **Entering variable:**

in std. we look at tableau, in revised we need to compute:  $\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N$

1. find  $\mathbf{y}^T = \mathbf{c}_B^T A_B^{-1}$  (by solving  $\mathbf{y}^T A_B = \mathbf{c}_B^T$ , the latter can be done more efficiently)
2. calculate  $\mathbf{c}_N^T - \mathbf{y}^T A_N$

Step 1:

$$[y_1 \ y_2 \ y_3] \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = [1 \ 1 \ 0]$$

$$\mathbf{y}^T A_B = \mathbf{c}_B^T$$

$$[1 \ 1 \ 0] \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = [-1 \ 0 \ 2]$$

$$\mathbf{c}_B^T A_B^{-1} = \mathbf{y}^T$$

Step 2:

$$[0 \ 0] - [-1 \ 0 \ 2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [1 \ -2]$$

$$\mathbf{c}_N^T - \mathbf{y}^T A_N$$

(Note that they can be computed individually:  $c_j - \mathbf{y}^T \mathbf{a}_j > 0$ )

Let's take the first we encounter  $x_3$

- **Leaving variable**

we increase variable by largest feasible amount  $\theta$

$$\text{R1: } x_1 - x_3 + x_5 = 1$$

$$x_1 = 1 + x_3 \geq 0$$

$$\text{R2: } x_2 + 0x_3 + x_5 = 2$$

$$x_2 = 2 \geq 0$$

$$\text{R3: } -x_3 + x_4 - x_5 = 2$$

$$x_4 = 2 - x_3 \geq 0$$

$$\mathbf{x}_B = \mathbf{x}_B^* - A_B^{-1} A_N \mathbf{x}_N$$

$$\mathbf{x}_B = \mathbf{x}_B^* - \mathbf{d}\theta$$

$\mathbf{d}$  is the column of  $A_B^{-1} A_N$  that corresponds to the entering variable, ie,  $\mathbf{d} = A_B^{-1} \mathbf{a}$  where  $\mathbf{a}$  is the entering column

3. Find  $\theta$  such that  $\mathbf{x}_B$  stays positive:

Find  $\mathbf{d} = A_B^{-1} \mathbf{a}$  (by solving  $A_B \mathbf{d} = \mathbf{a}$ )

Step 3:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \mathbf{d} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \implies \mathbf{x}_B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \theta \geq 0$$

$$2 - \theta \geq 0 \implies \theta \leq 2 \rightsquigarrow x_4 \text{ leaves}$$

- So far we have done computations, but now we save the pivoting update. The update of  $A_B$  is done by replacing the leaving column by the entering column

$$x_B^* = \begin{bmatrix} x_1 - d_1\theta \\ x_2 - d_2\theta \\ \theta \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \quad A_B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Many implementations depending on how  $\mathbf{y}^T A_B = \mathbf{c}_B^T$  and  $A_B \mathbf{d} = \mathbf{a}$  are solved. They are in fact solved from scratch.
- many operations saved especially if many variables!
- special ways to call the matrix  $A$  from memory
- better control over numerical issues since  $A_B^{-1}$  can be recomputed.

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# Solving the two Systems of Equations

$A_B \mathbf{x} = \mathbf{b}$  solved without computing  $A_B^{-1}$   
(costly and likely to introduce numerical inaccuracy)

Recall how the inverse is computed:

For a  $2 \times 2$  matrix

the matrix inverse is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a  $3 \times 3$  matrix

the matrix inverse is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$

# Eta Factorization of the Basis

Let  $B := A_B$ ,  $k$ th iteration

$B_k$  be the matrix with col  $p$  differing from  $B_{k-1}$

Column  $p$  is the  $\mathbf{a}$  column appearing in  $B_{k-1}\mathbf{d} = \mathbf{a}$  solved at 3)

Hence:

$$B_k = B_{k-1}E_k$$

$E_k$  is the **eta matrix** differing from id. matrix in only one column

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

No matter how we solve  $\mathbf{y}^T B_{k-1} = \mathbf{c}_B^T$  and  $B_{k-1}\mathbf{d} = \mathbf{a}$ , their update always relies on  $B_k = B_{k-1}E_k$  with  $E_k$  available.

Plus when initial basis by slack variable  $B_0 = I$  and  $B_1 = E_1, B_2 = E_1E_2 \dots$ :

$$B_k = E_1E_2 \dots E_k \quad \text{eta factorization}$$

$$\begin{aligned} ((((\mathbf{y}^T E_1)E_2)E_3) \dots)E_k &= \mathbf{c}_B^T, \\ (E_1(E_2 \dots E_k \mathbf{d})) &= \mathbf{a}, \end{aligned}$$

$$\begin{aligned} \mathbf{u}^T E_4 &= \mathbf{c}_B^T, \quad \mathbf{v}^T E_3 = \mathbf{u}^T, \quad \mathbf{w}^T E_2 = \mathbf{v}^T, \quad \mathbf{y}^T E_1 = \mathbf{w}^T \\ E_1 \mathbf{u} &= \mathbf{a}, \quad E_2 \mathbf{v} = \mathbf{u}, \quad E_3 \mathbf{w} = \mathbf{v}, \quad E_4 \mathbf{d} = \mathbf{w} \end{aligned}$$

# Exercise

Solve the systems  $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1 \ 2 \ 3]$  and  $E_1 E_2 E_3 E_4 \mathbf{d} = [1 \ 2 \ 3]^T$  with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$



We use backward transformation and solve the sequence of linear systems:

$$\mathbf{u}^T E_4 = [1 \ 2 \ 3], \quad \mathbf{v}^T E_3 = \mathbf{u}^T, \quad \mathbf{w}^T E_2 = \mathbf{v}^T, \quad \mathbf{y}^T E_1 = \mathbf{w}^T$$

$$\mathbf{u}^T \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [1, 2, 3]$$

Since the eta matrices have always one 1 in two columns then the solution can be read up easily. From the third column we find  $u_3 = 3$ . From the second column, we find  $u_2 = 2$ . Substituting in the first column, we find  $-0.5u_1 + 3 * 2 + 1 * 3 = 1$ , which yields  $u_1 = 18$ . The next system is:

$$\mathbf{v}^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [18, 2, 3]$$

From the first column we get  $v_1 = 18$ , from the second column  $v_2 = 2$  from the last column  $v_3 = 3/24$ . The next:

$$\mathbf{w} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = [18, 2, 3/24]$$

- Solving  $\mathbf{y}^T B_k = \mathbf{c}_B^T$  also called backward transformation (BTRAN)
- Solving  $B_k \mathbf{d} = \mathbf{a}$  also called forward transformation (FTRAN)
- $E_i$  matrices can be stored by only storing the column and the position
- If sparse columns then can be stored in compact mode, ie only nonzero values and their indices

- Tableau method is unstable: computational errors may accumulate. Revised method has a natural control mechanism: we can recompute  $A_B^{-1}$  at any time
- Commercial and freeware solvers differ from the way the systems  $\mathbf{y}^T A_B = \mathbf{c}_B^T$  and  $A_B \mathbf{d} = \mathbf{a}$  are resolved

# Efficient Implementations

- Dual simplex with steepest descent (largest increase)
- Linear Algebra:
  - Dynamic LU-factorization using Markowitz threshold pivoting (Suhl and Suhl, 1990)
  - sparse linear systems: Typically these systems take as input a vector with a very small number of nonzero entries and output a vector with only a few additional nonzeros.
- Presolve, ie problem reductions: removal of redundant constraints, fixed variables, and other extraneous model elements.
- dealing with degeneracy, stalling (long sequences of degenerate pivots), and cycling:
  - bound-shifting (Paula Harris, 1974)
  - Hybrid Pricing (variable selection): start with partial pricing, then switch to devex (approximate steepest-edge, Harris, 1974)
- A model that might have taken a year to solve 10 years ago can now solve in less than 30 seconds (Bixby, 2002).