DM545/DM871

Linear and Integer Programming

Lecture 9

Integer Linear Programming Modeling

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Outline

1. Integer Programming

2. Modeling
   - Assignment Problem
   - Knapsack Problem
   - Set Problems

3. More on Modeling
   - Graph Problems
   - Modeling Tricks

4. Formulations
   - Uncapacited Facility Location
   - Alternative Formulations

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Discrete Optimization

- Often we need to deal with integral inseparable quantities
- Sometimes rounding can go
- Other times rounding not feasible: eg, presence of a bus on a line is 0.3...

Integer Linear Programming

Linear Objective
Linear Constraints
but! integer variables

The world is not linear: "OR is the art and science of obtaining bad answers to questions to which otherwise worse answers would be given"

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Linear Programming (LP)

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{x} &\ \text{ integer} \end{aligned}$$

Integer (Linear) Programming (ILP)

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\in \{0,1\}^n \end{aligned}$$

Binary Integer Program (BIP)
0/1 Integer Programming

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} + \mathbf{h}^T\mathbf{y} \\ A\mathbf{x} + \ G\mathbf{y} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{y} &\geq \mathbf{0} \\ \mathbf{y} &\ \text{ integer} \end{aligned}$$

Mixed Integer (Linear) Programming (MILP)

$$\begin{aligned} \max\ & f(\mathbf{x}) \\ g(\mathbf{x}) &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Non-linear Programming (NLP)

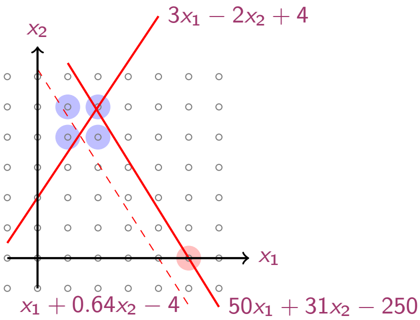
Recall:

- $\mathbb{Z}$ set of integers
- $\mathbb{Z}^+$ set of positive integer
- $\mathbb{Z}_0^+$ set of nonnegative integers ($\{0\} \cup \mathbb{Z}^+$)
- $\mathbb{N}_0$ set of natural numbers, ie, nonnegative integers $\{0, 1, 2, 3, 4, \ldots\}$

Rounding

$$\begin{aligned} \max\ 100x_1 + 64x_2 & \\ 50x_1 + 31x_2 &\leq 250 \\ 3x_1 - \ \ 2x_2 &\geq -4 \\ x_1, x_2 &\in \mathbb{Z}_0^+ \end{aligned}$$

LP optimum $(376/193, 950/193)$
IP optimum $(5, 0)$



Note: rounding does not help in the example above!

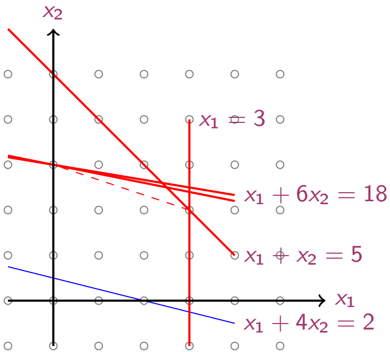
$\rightsquigarrow$ feasible region convex but not continuous: Now the optimum can be on the border (vertices) but also internal.

Possible way: solve the relaxed problem.

- If solution is integer, done.
- If solution is rational (never irrational) try rounding to the nearest integers (but may exit feasibility region)
  if in $\mathbb{R}^2$ then 2^2 possible roundings (up or down)
  if in $\mathbb{R}^n$ then 2^n possible roundings (up or down)

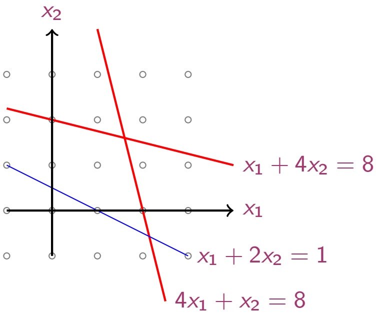
Cutting Planes

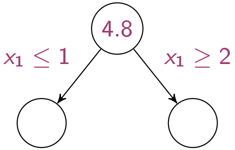
$$\begin{aligned} \max\ x_1 + 4x_2 & \\ x_1 + 6x_2 &\leq 18 \\ x_1 \quad\quad &\leq 3 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 & \text{ integers} \end{aligned}$$

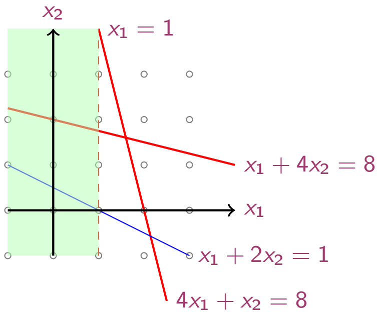


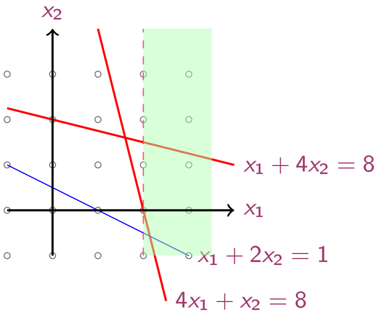
Branch and Bound

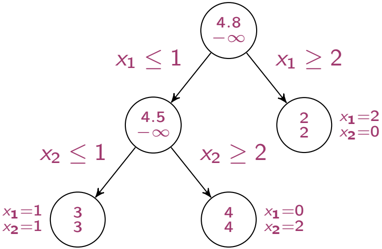
$$\begin{aligned} \max\ & x_1 + 2x_2 \\ & x_1 + 4x_2 \leq 8 \\ & 4x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0, \text{integer} \end{aligned}$$



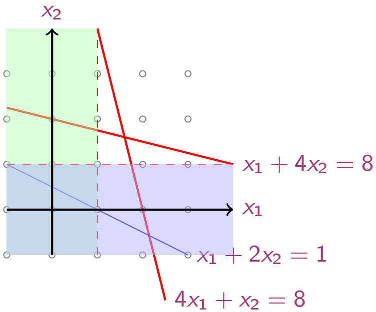

4.8
$x_1 \leq 1$
$x_1 \geq 2$


x_2
$x_1 = 1$
$x_1 + 4x_2 = 8$
x_1
$x_1 + 2x_2 = 1$
$4x_1 + x_2 = 8$

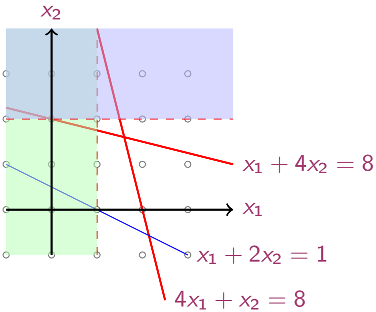

x_2
$x_1 + 4x_2 = 8$
x_1
$x_1 + 2x_2 = 1$
$4x_1 + x_2 = 8$



4.8
$-\infty$
$x_1 \leq 1$
$x_1 \geq 2$
4.5
$-\infty$
2
2
x_1=2
x_2=0
$x_2 \leq 1$
$x_2 \geq 2$
x_1=1
x_2=1
3
3
4
4
x_1=0
x_2=2



x_2
$x_1 + 4x_2 = 8$
x_1
$x_1 + 2x_2 = 1$
$4x_1 + x_2 = 8$



x_2
$x_1 + 4x_2 = 8$
x_1
$x_1 + 2x_2 = 1$
$4x_1 + x_2 = 8$

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Mathematical Programming: Modeling

- Find out exactly what the decision maker needs to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Decision Variables of suitable type (continuous, integer valued, binary) corresponding to the needs and Known Parameters corresponding to given data.
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

How to “build” a constraint

- Formulate relationship between the variables in plain words
- Then formulate your sentences using logical connectives **and, or, not, implies**
- Finally convert the logical statement to a mathematical constraint.

Example

- “The power plant must not work in both of two neighbouring time periods”
- on/off is modelled using **binary** integer variables
- $x_i = 1$ or $x_i = 0$
- $x_i = 1$ implies $\Rightarrow x_{i+1} = 0$
- $x_i + x_{i+1} \leq 1$

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The Assignment Problem

Problem

Common application: **Assignees** are being assigned to perform **tasks**.

Suppose we have n persons and n jobs
Each person has a certain proficiency at each job.

Formulate a mathematical model that can be used to find an assignment that maximizes the total proficiency.

Decision Variables:

$$x_{ij} = \begin{cases} 1 \text{ if person } i \text{ is assigned job } j \\ 0 \text{ otherwise,} \end{cases} \quad \text{for } i, j = 1, 2, \ldots, n$$

Objective Function:

$$\max \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} x_{ij}$$

where ρ_{ij} is person i's proficiency at job j

Constraints:

Each person is assigned one job:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i$$

e.g. for person 1 we get $x_{11} + x_{12} + x_{13} + \cdots + x_{1n} = 1$

Each job is assigned to one person:

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for all } j$$

e.g. for job 1 we get $x_{11} + x_{21} + x_{31} + \cdots + x_{n1} = 1$

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The Knapsack Problem

Problem ..

Input: Given a set of n items, each with a value v_i and weight w_i $(i = 1, \ldots, n)$

Task: determine the number of each items to include in a collection so that the total weight is less than a given limit, W, and the total value is as large as possible.

The "knapsack" name derives from the problem faced by someone who is constrained by a fixed-size knapsack and must fill it with the most useful items.

Assuming we can take at most one of any item and that $\sum_i w_i > W$, formulate a mathematical model to determine which items give the largest value.

Model used, eg, in capital budgeting, project selection, etc.

The Knapsack Problem

Decision Variables:

$$x_i = \begin{cases} 1 \text{ if item } i \text{ is taken} \\ 0 \text{ otherwise,} \end{cases} \quad \text{for } i = 1, 2 \ldots, n$$

Objective Function:

$$\max \quad \sum_{i=1}^{n} v_i x_i$$

Constraints:
Knapsack capacity restriction:

$$\sum_{i=1}^{n} w_i x_i \leq W$$

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Set Covering

Problem

Given: a set of regions, a set of possible construction locations for emergency centers, regions that can be served in less than 8 minutes, cost of installing an emergency center in each location.

Task: decide where to install a set of emergency centers such that the total cost is minimized and all regions are safely served

As a COP: $M = \{1, \dots, m\}$ regions, $N = \{1, \dots, n\}$ centers, $S_j \subseteq M$ regions serviced by $j \in N$ in 8 min.

$$\min_{T \subseteq N} \left\{ \sum_{j \in T} c_j \mid \bigcup_{j \in T} S_j = M \right\}$$

regions: $M = \{1, \dots, 5\}$
centers: $N = \{1, \dots, 6\}$
cost of centers: $c_j = 1 \quad \forall j = 1, \dots, 6$
coverages: $S_1 = (1,2), S_2 = (1,3,5), S_3 = (2,4,5), S_4 = (3), S_5 = (1), S_6 = (4,5)$

Example

- regions: $M = \{1, \ldots, 5\}$
  centers: $N = \{1, \ldots, 6\}$
  cost of centers: $c_j = 1 \quad \forall j = 1, \ldots, 6$
  coverages: $S_1 = (1, 2), S_2 = (1, 3, 5), S_3 = (2, 4, 5), S_4 = (3), S_5 = (1), S_6 = (4, 5)$
-

$$
A = \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

As a BIP:

Variables:

$\mathbf{x} \in \mathbb{B}^n$, $x_j = 1$ if center j is selected, 0 otherwise

Objective:

$$\min \sum_{j=1}^{n} c_j x_j$$

Constraints:

- incidence matrix: $a_{ij} = \begin{cases} 1 \\ 0 \end{cases}$
- $\sum_{j=1}^{n} a_{ij} x_j \geq 1$

Set covering

cover each of M at least once

1. min, $\geq$
2. all RHS terms are 1
3. all matrix elements are 1

$$\begin{aligned} \min \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} \geq \mathbf{1} \\ \mathbf{x} \in \mathbb{B}^n \end{aligned}$$

Set packing

cover as many of M without overlap

1. max, $\leq$
2. all RHS terms are 1
3. all matrix elements are 1

$$\begin{aligned} \max \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} \leq \mathbf{1} \\ \mathbf{x} \in \mathbb{B}^n \end{aligned}$$

Set partitioning

cover exactly once each element of M

1. max or min, $=$
2. all RHS terms are 1
3. all matrix elements are 1

$$\begin{aligned} \max \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} = \mathbf{1} \\ \mathbf{x} \in \mathbb{B}^n \end{aligned}$$

Generalization: $RHS \geq 1$
Application examples:

- Aircrew scheduling: M: legs to cover, N: rosters
- Vehicle routing: M: customers, N: routes

A good written example of how to present a model:

2.1. Notation

Let N be the set of operational flight legs and K the set of aircraft types. Denote by n^k the number of available aircraft of type $k \in K$. Define Ω^k, indexed by p, as the set of feasible schedules for aircraft of type $k \in K$ and let index $p = 0$ denote the empty schedule for an aircraft. Next associate with each schedule $p \in \Omega^k$ the value c_p^k denoting the anticipated profit if this schedule is assigned to an aircraft of type $k \in K$ and a_{ip}^k a binary constant equal to 1 if this schedule covers flight leg $i \in N$ and 0 otherwise. Furthermore, let S be the set of stations and $S^k \subseteq S$ the subset having the facilities to serve aircraft of type $k \in K$. Then, define o_{sp}^k and d_{sp}^k to equal to 1 if schedule p, $p \in \Omega^k$, starts and ends respectively at station s, $s \in S^k$, and 0 otherwise.

Denote by θ_p^k, $p \in \Omega^k \backslash \{0\}$, $k \in K$, the binary decision variable which takes the value 1 if schedule p is assigned to an aircraft of type k, and 0 otherwise. Finally, let θ_0^k, $k \in K$, be a nonnegative integer variable which gives the number of unused aircraft of type k.

2.2. Formulation

Using these definitions, the DARSP can be formulated as:

$$\text{Maximize} \sum_{k \in K} \sum_{p \in \Omega^k} c_p^k \theta_p^k \tag{1}$$

subject to:

$$\sum_{k \in K} \sum_{p \in \Omega^k} a_{ip}^k \theta_p^k = 1 \quad \forall i \in N, \tag{2}$$

$$\sum_{p \in \Omega^k} (d_{sp}^k - o_{sp}^k)\theta_p^k = 0 \quad \forall k \in K, \forall s \in S^k, \tag{3}$$

$$\sum_{p \in \Omega^k} \theta_p^k = n^k \quad \forall k \in K, \tag{4}$$

$$\theta_p^k \geq 0 \quad \forall k \in K, \forall p \in \Omega^k, \tag{5}$$

$$\theta_p^k \text{ integer} \quad \forall k \in K, \forall p \in \Omega^k. \tag{6}$$

The objective function (1) states that we wish to maximize the total anticipated profit. Constraints (2) require that each operational flight leg be covered exactly once. Constraints (3) correspond to the flow conservation constraints at the beginning and the end of the day at each station and for each aircraft type. Constraints (4) limit the number of aircraft of type $k \in K$ that can be used to the number available. Finally, constraints (5) and (6) state that the decision variables are nonnegative integers. This model is a Set Partitioning problem with additional constraints.

[from G. Desaulniers, J. Desrosiers, Y. Dumas, M.M. Solomon and F. Soumis. Daily Aircraft Routing and Scheduling. Management Science, 1997, 43(6), 841-855]

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Matching

Definition (Matching Theory Terminology)

Matching: set of pairwise non adjacent edges

Covered (vertex): a vertex is covered by a matching M if it is incident to an edge in M

Perfect (matching): if M covers each vertex in G

Maximal (matching): if M cannot be extended any further

Maximum (matching): if M covers as many vertices as possible

Matchable (graph): if the graph G has a perfect matching

$$
\begin{aligned}
\max \sum_{v \in V} w_e x_e & \\
\sum_{e \in E : v \in e} x_e \leq 1 & \quad \forall v \in V \\
x_e \in \{0,1\} & \quad \forall e \in E
\end{aligned}
$$

Special case: bipartite matching $\equiv$ assignment problems

Vertex Cover

Select a subset $S \subseteq V$ such that each edge has at least one end vertex in S.

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ x_v + x_u \geq 1 & \qquad \forall u, v \in V, uv \in E \\ x_v \in \{0,1\} & \; \forall v \in V \end{aligned}$$

Approximation algorithm: set S derived from the LP solution in this way:

$$S_{LP} = \{v \in V : x_v^* \geq 1/2\}$$

(it is a cover since $x_v^* + x_u^* \geq 1$ implies $x_v^* \geq 1/2$ or $x_u^* \geq 1/2$)

Proposition

The LP rounding approximation algorithm gives a 2-approximation: $|S_{LP}| \leq 2|S_{OPT}|$ *(at most as bad as twice the optimal solution)*

Proof: Let $\bar{x}$ be opt to IP. Then $\sum x_v^* \leq \sum \bar{x}_v$.
$|S_{LP}| = \sum_{v \in S_{LP}} 1 \leq \sum_{v \in V} 2x_v^*$ since $x_v^* \geq 1/2$ for each $v \in S_{LP}$
$|S_{LP}| \leq 2 \sum_{v \in V} x_v^* \leq 2 \sum_{v \in V} \bar{x}_v = 2|S_{OPT}|$

Maximum Independent Set

Find the largest subset $S \subseteq V$ such that the induced graph has no edges

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ x_v + x_u &\leq 1 \qquad \forall u, v \in V, uv \in E \\ x_v &= \{0, 1\} \; \forall v \in V \end{aligned}$$

Optimal sol of LP relaxation sets $x_v = 1/2$ for all variables and has value $|V|/2$.

What is the value of an optimal IP solution of a complete graph?

LP relaxation gives an $O(n)$-approximation (almost useless)

Traveling Salesman Problem

- Find the cheapest movement for a drilling, welding, drawing, soldering arm as, for example, in a printed circuit board manufacturing process or car manufacturing process
- n locations, c_{ij} cost of travel

Variables:

$$x_{ij} = \begin{cases} 1 \\ 0 \end{cases}$$

Objective:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Constraints:

-

$$\sum_{j:j\neq i} x_{ij} = 1 \qquad \forall i = 1, \ldots, n$$

$$\sum_{i:i\neq j} x_{ij} = 1 \qquad \forall j = 1, \ldots, n$$

- cut set constraints

$$\sum_{i\in S} \sum_{j\notin S} x_{ij} \geq 1 \qquad \forall S \subset N, S \neq \emptyset$$

- subtour elimination constraints

$$\sum_{i\in S} \sum_{j\in S} x_{ij} \leq |S| - 1 \qquad \forall S \subset N, 2 \leq |S| \leq n - 1$$

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Modeling Tricks

Objective function and/or constraints do not appear to be linear?

- Absolute values
- Minimize the largest function value
- Maximize the smallest function value
- Constraints include variable division
- Constraints are either/or
- A variable must take one of several candidate values

Modeling: Absolute Values

$$\min \sum_{i=1}^{n} |f_i(\mathbf{x})|$$

$$\begin{aligned}
\min\ & \textstyle\sum_{i=1}^{n} z_i \\
\text{s.t.}\ & z_i \geq f_i(\mathbf{x}) && i = 1..n \\
& z_i \geq -f_i(\mathbf{x}) && i = 1..n \\
& z_i \in \mathbb{R} && i = 1..n \\
& \mathbf{x} \in \mathbb{R}^q
\end{aligned}$$

n additional variables and $2n$ additional constraints.

$$\begin{aligned}
\min\ & \textstyle\sum_{i=1}^{n} (z_i^+ + z_i^-) \\
\text{s.t.}\ & f_i(\mathbf{x}) = z_i^+ - z_i^- && i = 1..n \\
& z_i^+, z_i^- \geq 0 && i = 1..n \\
& \mathbf{x} \in \mathbb{R}^q
\end{aligned}$$

$2n$ additional variables and n additional constraints.

Modeling: Minimax

Minimize the largest of a number of function values:

$$\min \quad \max\{f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\}$$

- Introduce an auxiliary variable z:

$$\begin{aligned} \min \quad & z \\ \text{s. t.} \ & f_1(\mathbf{x}) \leq z \\ & f_2(\mathbf{x}) \leq z \end{aligned}$$

Modeling: Divisions

Constraints include variable division:

- Constraint of the form

$$\frac{a_1 x + a_2 y + a_3 z}{d_1 x + d_2 y + d_3 z} \leq b$$

- Rearrange:

$$a_1 x + a_2 y + a_3 z \leq b(d_1 x + d_2 y + d_3 z)$$

which gives:

$$(a_1 - bd_1)x + (a_2 - bd_2)y + (a_3 - bd_3)z \leq 0$$

Modeling: "Either/Or Constraints"

In conventional mathematical models, the solution must satisfy all constraints. Suppose that your constraints are "either/or":

$$a_1x_1 + a_2x_2 \leq b_1 \qquad \text{or}$$
$$d_1x_1 + d_2x_2 \leq b_2$$

Introduce new variable $y \in \{0, 1\}$ and a large number M:

$$a_1x_1 + a_2x_2 \leq b_1 + My \qquad \text{if } y = 0 \text{ then this is active}$$
$$d_1x_1 + d_2x_2 \leq b_2 + M(1 - y) \qquad \text{if } y = 1 \text{ then this is active}$$

Modeling: “Either/Or Constraints”

Binary integer programming allows to model alternative choices:

- Eg: 2 feasible regions, ie, disjunctive constraints, not possible in LP.
  introduce y auxiliary binary variable and M a big number:

$$\begin{aligned} Ax &\leq b + My && \text{if } y = 0 \text{ then this is active} \\ A'x &\leq b' + M(1 - y) && \text{if } y = 1 \text{ then this is active} \end{aligned}$$

Modeling: "Either/Or Constraints"

Generally:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1m}x_m &\leq d_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2m}x_m &\leq d_2 \\
&\vdots \\
a_{m1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{Nm}x_m &\leq d_N
\end{aligned}
$$

Exactly K of the N constraints must be satisfied.
Introduce binary variables $y_1, y_2, \ldots, y_N$ and a large number M

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1m}x_m &\leq d_1 + My_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2m}x_m &\leq d_2 + My_2 \\
&\vdots \\
a_{m1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{Nm}x_m &\leq d_N + My_N
\end{aligned}
$$

$$y_1 + y_2 + \ldots y_N = N - K$$

K of the y-variables are 0, so K constraints must be satisfied

Modeling: "Either/Or Constraints"

At least $h \leq k$ of $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, \ldots, k$ must be satisfied

introduce y_i, $i = 1, ..., k$ auxiliary binary variables

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + My_i$$

$$\sum_{i} y_i \leq k - h$$

Modeling: "Possible Constraints Values"

A constraint must take on one of N given values:

$$
\begin{aligned}
a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m &= d_1 \text{ or}\\
a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m &= d_2 \text{ or}\\
&\vdots\\
a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m &= d_N
\end{aligned}
$$

Introduce binary variables $y_1, y_2, \ldots, y_N$:

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_mx_m = d_1y_1 + d_2y_2 + \ldots d_Ny_N$$

$$y_1 + y_2 + \ldots y_N = 1$$

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4. Formulations
   - Uncapacited Facility Location
   - Alternative Formulations

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Uncapacited Facility Location (UFL)

Given:

- depots $N = \{1, \dots, n\}$
- clients $M = \{1, \dots, m\}$
- f_j fixed cost to use depot j
- transport cost for all orders c_{ij}

Task: Which depots to open and which depots serve which client

Variables: $y_j = \begin{cases} 1 & \text{if depot opened} \\ 0 & \text{otherwise} \end{cases}$, x_{ij} fraction of demand of i satisfied by j

Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

Constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \dots, m$$

$$\sum_{i \in M} x_{ij} \leq m y_j \qquad \forall j \in N$$

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Good and Ideal Formulations

Definition (Formulation)

A polyhedron $P \subseteq \mathbb{R}^{n+p}$ is a formulation for a set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if and only if $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region X.

There are infinite formulations

Definition (Convex Hull)

Given a set $X \subseteq \mathbb{Z}^n$ the convex hull of X is defined as:

$$\text{conv}(X) = \Big\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^{t} \lambda_i \mathbf{x}^i, \qquad \sum_{i=1}^{t} \lambda_i = 1, \qquad \lambda_i \geq 0, \qquad \text{for } i = 1, \ldots, t, \\ \text{for all finite subsets } \{\mathbf{x}^1, \ldots, \mathbf{x}^t\} \text{ of } X \Big\}$$

Proposition

$conv(X)$ is a polyhedron (ie, representable as $A\mathbf{x} \leq \mathbf{b}$)

Proposition

Extreme points of $conv(X)$ all lie in X

Hence:

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\} \equiv \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \text{conv}(X)\}$$

However it might require exponential number of inequalities to describe $\text{conv}(X)$
What makes a formulation better than another?

$$X \subseteq \text{conv}(X) \subseteq P_2 \subset P_1$$

$$P_2 \text{ is better than } P_1$$

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, P_2 is a better formulation than P_1 if $P_2 \subset P_1$

Example

$P_1 =$ UFL with $\sum_{i \in M} x_{ij} \leq m y_j \quad \forall j \in N$
$P_2 =$ UFL with $x_{ij} \leq y_j \quad \forall i \in M, j \in N$

$$P_2 \subset P_1$$

- $P_2 \subseteq P_1$ because summing $x_{ij} \leq y_j$ over $i \in M$ we obtain $\sum_{i \in M} x_{ij} \leq m y_j$
- $P_2 \subset P_1$ because there exists a point in P_1 but not in P_2: $m = 6 = 3 \cdot 2 = k \cdot n$

$$x_{10} = 1,\ x_{20} = 1,\ x_{30} = 1, \qquad \sum_i x_{i0} \leq 6 y_0 \quad y_0 = 1/2$$
$$x_{41} = 1,\ x_{51} = 1,\ x_{61} = 1 \qquad \sum_i x_{i1} \leq 6 y_1 \quad y_1 = 1/2$$

Resume

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