DM545/DM871

Linear and Integer Programming

Lecture 3

The Simplex Method

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Outline

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A Numerical Example

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ i = 1, \ldots, m \\ & x_j \geq 0, \ j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max \ 6x_1 &+ 8x_2 \\ 5x_1 &+ 10x_2 \leq 60 \\ 4x_1 &+ 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \ & \mathbf{c}^T \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$

$$\max \ \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

Standard Form

Every LP problem can be converted in the standard form:

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\in \mathbb{R}^n \end{aligned}$$

$$\mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$$

- if equations, then put two constraints, $\mathbf{a}x \leq b$ and $\mathbf{a}x \geq b$
- if $\mathbf{a}x \geq b$ then $-\mathbf{a}x \leq -b$
- if $\min \mathbf{c}^T\mathbf{x}$ then $\max(-\mathbf{c}^T\mathbf{x})$

and then be put in equational standard form:

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$$

1. "$=$" constraints
2. $\mathbf{x} \geq 0$ nonnegativity constraints
3. $(\mathbf{b} \geq 0)$
4. $\max$

Transformation to Std Form

Every LP problem can be transformed in eq. std. form

1. introduce slack variables (or surplus)

$$\begin{aligned} 5x_1 + 10x_2 + x_3 &= 60 \\ 4x_1 + 4x_2 + x_4 &= 40 \end{aligned}$$

2. if $x_1 \gtreqless 0$ then $\begin{aligned} x_1 &= x_1' - x_1'' \\ x_1' &\geq 0 \\ x_1'' &\geq 0 \end{aligned}$

3. $(b \geq 0)$

4. $\min c^T x \equiv \max(-c^T x)$

LP in $m \times n$ converted into LP with at most $(m + 2n)$ variables and m equations (n # original variables, m # constraints)

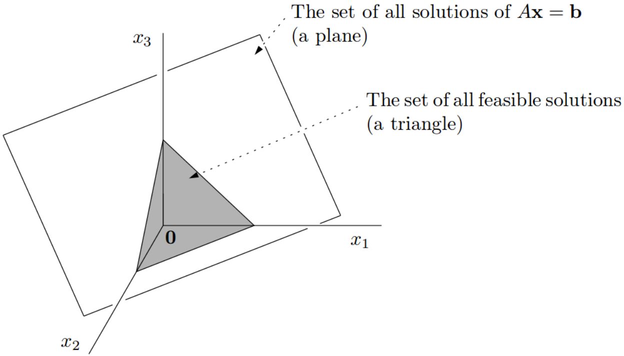
Geometry of LP in Eq. Std. Form

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

From linear algebra:

- the set of solutions of $A\mathbf{x} = \mathbf{b}$ is an affine space (hyperplane not passing through the origin).
- $\mathbf{x} \geq \mathbf{0}$ nonegative orthant (octant in $\mathbb{R}^3$)

In $\mathbb{R}^3$:



- $A\mathbf{x} = \mathbf{b}$ is a system of equations that we can solve by Gaussian elimination

- Elementary row operations of $\left[A \mid \mathbf{b}\right]$ do not affect set of feasible solutions
  - multiplying all entries in some row of $\left[A \mid \mathbf{b}\right]$ by a nonzero real number λ
  - replacing the ith row of $\left[A \mid b\right]$ by the sum of the ith row and jth row for some $i \neq j$

- Let n' be the number of vars in eq. std. form.

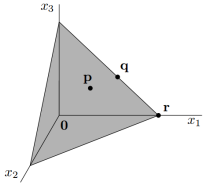
  we assume $n' \geq m$ and $\mathrm{rank}(\left[A \mid \mathbf{b}\right]) = \mathrm{rank}(A) = m$

  ie, rows of A are linearly independent
  otherwise, remove linear dependent rows

Outline

Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:
Let $B = \{1 \dots m\}$, $N = \{m+1 \dots n+m = n'\}$ be subsets partitioning the columns of A: A_B be made of columns of A indexed by B:

Definition

$\mathbf{x} \in \mathbb{R}^n$ is a basic feasible solution of the linear program $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ for an index set B if:

- $x_j = 0 \; \forall j \notin B$
- the square matrix A_B is nonsingular, ie, all columns indexed by B are lin. indep.
- $\mathbf{x}_B = A_B^{-1}\mathbf{b}$ is nonnegative, ie, $\mathbf{x}_B \geq 0$ (feasibility)

We call x_j for $j \in B$ basic variables and remaining variables nonbasic variables.

Theorem

A basic feasible solution is uniquely determined by the set B.

Proof:

$$
\begin{aligned}
A\mathbf{x} = & A_B \mathbf{x}_B + A_N \mathbf{x}_N = b \\
& \mathbf{x}_B + A_B^{-1} A_N \mathbf{x}_N = A_B^{-1} b \\
& \mathbf{x}_B = A_B^{-1} b
\end{aligned}
$$

A_B is nonsingular hence one solution

Note: we call B a (feasible) basis

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

Theorem

Let P be a (convex) polyhedron from LP in eq. std. form. For a point $v \in P$ the following are equivalent:

(i) *v is an extreme point (vertex) of P*

(ii) *v is a basic feasible solution of LP*

Proof: see text book [MG] sec. 4.4.

Theorem

Let $LP = \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be feasible and bounded, then the optimal solution is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming

Note, a similar theorem is valid for arbitrary linear programs (not in eq. form)

Definition

A basic feasible solution of a linear program with n variables is a feasible solution for which some n linearly independent constraints hold with equality.

However, an optimal solution does not need to be basic:

$$\max x_1 + x_2 \text{ subject to } x_1 + x_2 \leq 1$$

- Idea for solution method:
- examine all basic solutions.
- There are finitely many: $\binom{m+n}{m}$.
- However, if $n = m$ then $\binom{2m}{m} \approx 4^m$.

Outline

Simplex Method

$$\max \quad z = \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Canonical eq. std. form: one decision variable is isolated in each constraint with coefficient 1 and does not appear in the other constraints nor in the obj. func. and b terms are positive

It gives immediately a basic feasible solution:

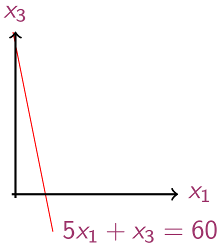
$$x_1 = 0,\ x_2 = 0,\ x_3 = 60,\ x_4 = 40$$

Is it optimal? Look at signs in $z \rightsquigarrow$ if positive then an increase would improve.

Let's try to increase a promising variable, ie, x_1, one with positive coefficient in z

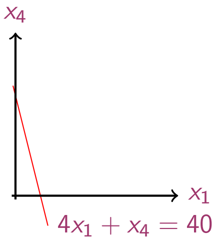
$$5x_1 + x_3 = 60$$
$$x_1 = \frac{60}{5} - \frac{x_3}{5}$$
$$x_3 = 60 - 5x_1 \geq 0$$

If $x_1 > 12$ then $x_3 < 0$



$$4x_1 + x_4 = 40$$
$$x_1 = \frac{40}{4} - \frac{x_4}{4}$$
$$x_4 = 40 - 4x_1 \geq 0$$

If $x_1 > 10$ then $x_4 < 0$



we can take the minimum of the two $\rightsquigarrow$ x_1 increased to 10
x_4 exits the basis and x_1 enters

Simplex Tableau

First simplex tableau:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

we want to reach this new tableau

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	?	1	?	0	?
x_1	1	?	0	?	0	?
	0	?	0	?	1	?

Pivot operation:

1. Choose pivot:

   column: one s with positive coefficient in obj. func.

   row: ratio between coefficient b and pivot column: choose the one with smallest ratio:

$$\theta = \min_i \left\{ \frac{b_i}{a_{is}} : a_{is} > 0 \right\},$$

θ increase value of entering var.

2. elementary row operations to update the tableau

- x_4 leaves the basis, x_1 enters the basis
  - Divide pivot row by pivot
  - Send to zero the coefficient in the pivot column of the first row
  - Send to zero the coefficient of the pivot column in the third (cost) row

	x1	x2	x3	x4	-z	b
I'=I-5II'	0	5	1	-5/4	0	10
II'=II/4	1	1	0	1/4	0	10
III'=III-6II'	0	2	0	-6/4	1	-60

From the last row we read: $2x_2 - 3/2x_4 - z = -60$, that is: $z = 60 + 2x_2 - 3/2x_4$.
Since x_2 and x_4 are nonbasic we have $z = 60$ and $x_1 = 10, x_2 = 0, x_3 = 10, x_4 = 0$.

- Done? No! Let x_2 enter the basis

	x1	x2	x3	x4	-z	b
I'=I/5	0	1	1/5	-1/4	0	2
II'=II-I'	1	0	-1/5	1/2	0	8
III'=III-2I'	0	0	-2/5	-1	1	-64

Definition (Reduced costs)

We call reduced costs the coefficients in the objective function of the nonbasic variables, $\bar{c}_N$

Proposition (Optimality Condition)

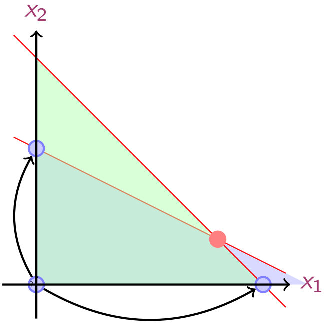
The basic feasible solution is optimal when the reduced costs in the corresponding simplex tableau are nonpositive, ie, such that:

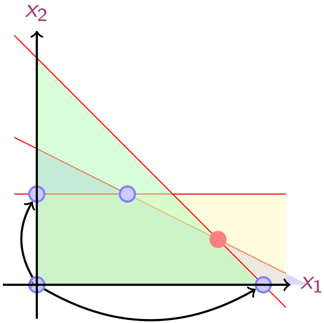
$$\bar{c}_N \leq 0$$

Proof: Let z_0 be the obj value when $\bar{c}_N \leq 0$.
For any other feasible solution $\tilde{\mathbf{x}}$ we have:

$$\tilde{\mathbf{x}}_N \geq 0 \quad \text{and} \quad \mathbf{c}^T\tilde{\mathbf{x}} = z_0 + \bar{\mathbf{c}}_N^T\tilde{\mathbf{x}}_N \leq z_0$$

Graphical Representation





Outline

Tableaux and Dictionaries

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ i = 1, \ldots, m \\ & x_j \geq 0, \ j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, m \\ z &= \sum_{j=1}^{n} c_j x_j \end{aligned}$$

Tableau

$$\left[\begin{array}{c|c|c|c} I & \bar{A}_N & 0 & \bar{b} \\ \hline 0 & \bar{c}_N & 1 & -\bar{d} \end{array}\right]$$

Dictionary

$$\begin{aligned} x_r &= \bar{b}_r - \sum_{s \notin B} \bar{a}_{rs} x_s, \quad r \in B \\ z &= \bar{d} + \sum_{s \notin B} \bar{c}_s x_s \end{aligned}$$

pivot operations in dictionary form:
choose col s with r.c. > 0
choose row with $\min\{-\bar{b}_i / \bar{a}_{is} \mid a_{is} < 0, i = 1, \ldots, m\}$
update: express entering variable and substitute in other rows

Example

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

$$\begin{array}{rcrcrcr} x_3 & = & 60 & - & 5x_1 & - & 10x_2 \\ x_4 & = & 40 & - & 4x_1 & - & 4x_2 \\ \hline z & = & & + & 6x_1 & + & 8x_2 \end{array}$$

After 2 iterations:

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	2
x_1	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

$$\begin{array}{rcrcrcr} x_2 & = & 2 & - & 1/5x_3 & + & 1/4x_4 \\ x_1 & = & 8 & + & 1/5x_3 & - & 1/2x_4 \\ \hline z & = & 64 & - & 2/5x_3 & - & 1x_4 \end{array}$$

Summary

1. Simplex Method
    - Standard Form
    - Basic Feasible Solutions
    - Algorithm
    - Tableaux and Dictionaries