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DM545/DM871 – Linear and integer programming

Exercise Sheet, Spring 2020 [pdf format]

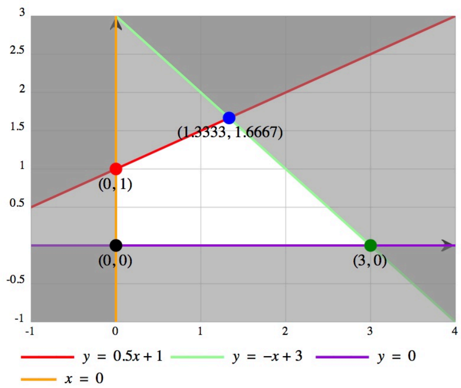
Solution:
Included.

Exercise 1*

Solve the following IP problem with Gomory's fractional cutting plane algorithm, indicating the cut inequalities in the space of the original variables

$$\begin{aligned} \max\ & x_1 + 2x_2 \\ & x_1 - 2x_2 \geq -2 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Solution:
We represent the situation graphically using the LP Grapher tool linked from the web page (any other graphing tool like Grapher under Macosx would do a similar job)




```
|-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    -z |     b |
|-------+-------+-------+-------+-------+-------+
|    -1 |     2 |     1 |     0 |     0 |     2 |
|     1 |     1 |     0 |     1 |     0 |     3 |
|-------+-------+-------+-------+-------+-------+
|     1 |     2 |     0 |     0 |     1 |     0 |
|-------+-------+-------+-------+-------+-------+
pivot column: 2
pivot row: 1
 pivot: 2

|-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    -z |     b |
|-------+-------+-------+-------+-------+-------+
|  -1/2 |     1 |   1/2 |     0 |     0 |     1 |
|   3/2 |     0 |  -1/2 |     1 |     0 |     2 |
```

2	0	-1	0	1	-2

```
pivot column: 1
pivot row: 2
 pivot: 3/2
```

x1	x2	x3	x4	-z	b
0	1	1/3	1/3	0	5/3
1	0	-1/3	2/3	0	4/3
0	0	-1/3	-4/3	1	-14/3

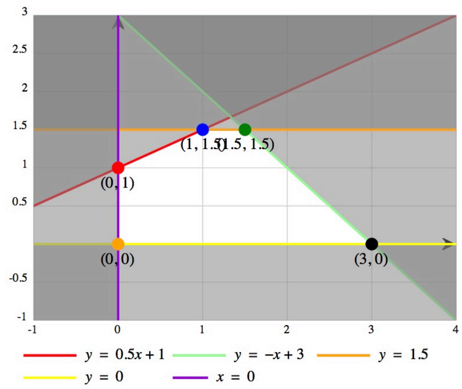
We choose the first row. The cut is:

$$1/3x_3 + 1/3x_4 \geq 2/3$$

In the original variables it is:

$$x_2 \leq 1$$

The cut separates the LP solution as evident from the picture below:



We insert the cut in the simplex and proceed by dual simplex:

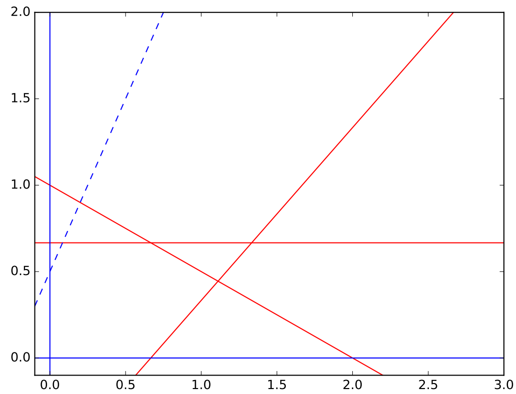
x1	x2	x3	x4	x5	-z	b
0	0	-1/3	-1/3	1	0	-2/3
0	1	1/3	1/3	0	0	5/3
1	0	-1/3	2/3	0	0	4/3
0	0	-1/3	-4/3	0	1	-14/3

Note that we could have inserted the cut also in the form $x_2 \leq 1$ but then we would have to put the tableau in canonical form.

Exercise 2 — Gomory's Cutting Plane

Consider the following integer linear programming problem

$$\begin{array}{llrcrl} \max & z = & 4x_0 & - & 2x_1 & \\ \text{s.t.} & & x_0 & + & 2x_1 & \leq 2 \\ & & & & 3x_1 & \leq 2 \\ & & 3x_0 & - & 3x_1 & \leq 2 \\ & & & & x_0, x_1 & \geq 0 \text{ and integer} \end{array}$$



In the solution of the linear relaxation of the problem the variables x_0, x_1 and the slack variable associated to the second constraint are in basis.

Subtask 2.1

Calculate the optimal tableau using the revised simplex method.

Solution:

Let x_2, x_3, x_4 be the slack variables in the equational standard form. Recalling the theory of the revised simplex, we can calculate the final table as follows:

$$\left[\begin{array}{c|c|c|c} I & \bar{A}_N = A_B^{-1}A_N & 0 & \bar{b} \\ \hline \bar{c}_B & \bar{c}_N(\leq 0) & 1 & -\bar{d} \end{array}\right]$$

Hence we need to determine $\bar{b}$ and $\bar{A}_N = A_B^{-1}A_N$. In Python:

```
AI = np.concatenate([A,np.identity(3)],axis=1)
ci=np.concatenate([c,[0,0,0,0]])

basis = np.array([0,1,3])
nonbasis = np.array([2,4])
B = AI[:,basis]
N = AI[:,nonbasis]



B_i = np.linalg.inv(B)

x_B = np.dot(B_i,b)
print(x_B)

y = np.dot(ci[basis],B_i)
red_costs = ci[nonbasis]-np.dot(y,N)
print( red_costs)


print(np.dot(ci[basis],x_B))

print(np.dot(B_i,N))

[ 1.111 0.444 0.667]
[-0.667 -1.111]
3.55555555556
[[ 0.333 0.222]
 [ 0.333 -0.111]
```

```
 [-1.  0.333]]
```

Hence the tableau looks as follows:

```
|----+----+--------+----+--------+----+--------|
| x0 | x1 |     x2 | x3 |     x4 | -z |      b |
|----+----+--------+----+--------+----+--------|
|  1 |  0 |  0.333 |  0 |  0.222 |  0 |  1.111 |
|  0 |  1 |  0.333 |  0 | -0.111 |  0 |  0.444 |
|  0 |  0 |     -1 |  1 |  0.333 |  0 |  0.667 |
|----+----+--------+----+--------+----+--------|
|  0 |  0 | -0.667 |  0 | -1.111 |  1 | -3.555 |
|----+----+--------+----+--------+----+--------|
```

Similarly, we could have used one of the available tools in fractional mode and obtained:

```
|--------+--------+--------+--------+--------+--------+--------+
|     x0 |     x1 |     x2 |     x3 |     x4 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+
|      1 |      2 |      1 |      0 |      0 |      0 |      2 |
|      0 |      3 |      0 |      1 |      0 |      0 |      2 |
|      3 |     -3 |      0 |      0 |      1 |      0 |      2 |
|--------+--------+--------+--------+--------+--------+--------+
|      4 |     -2 |      0 |      0 |      0 |      1 |      0 |
|--------+--------+--------+--------+--------+--------+--------+
PRIMAL SIMPLEX
Confirm pivot column: 00

pivot column: 1
pivot row: 3
 pivot: 3
3
|--------+--------+--------+--------+--------+--------+--------+
|     x0 |     x1 |     x2 |     x3 |     x4 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+
|      0 |      3 |      1 |      0 |   -1/3 |      0 |    4/3 |
|      0 |      3 |      0 |      1 |      0 |      0 |      2 |
|      1 |     -1 |      0 |      0 |    1/3 |      0 |    2/3 |
|--------+--------+--------+--------+--------+--------+--------+
|      0 |      2 |      0 |      0 |   -4/3 |      1 |   -8/3 |
|--------+--------+--------+--------+--------+--------+--------+
Confirm pivot column: 11

pivot column: 2
pivot row: 1
 pivot: 3
3
|--------+--------+--------+--------+--------+--------+--------+
|     x0 |     x1 |     x2 |     x3 |     x4 |     -z |      b |
|--------+--------+--------+--------+--------+--------+--------+
|      0 |      1 |    1/3 |      0 |   -1/9 |      0 |    4/9 |
|      0 |      0 |     -1 |      1 |    1/3 |      0 |    2/3 |
|      1 |      0 |    1/3 |      0 |    2/9 |      0 |   10/9 |
|--------+--------+--------+--------+--------+--------+--------+
|      0 |      0 |   -2/3 |      0 |  -10/9 |      1 |  -32/9 |
|--------+--------+--------+--------+--------+--------+--------+
```

Subtask 2.2

Find a Chvatal Gomory's cutting plane

Solution:

We consider the Gomory cut relative to the variable $x_0 = 1.111$.

$$\sum_{j \in N} (\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor) x_j \geq \bar{b}_u - \lfloor \bar{b}_u \rfloor$$

and substituting the numbers:

$$1/3x_2 + 2/9x_4 \geq 1/9$$

or

$$0.333x_2 + 0.222x_4 \geq 0.111$$

Subtask 2.3

Show that with the cut found the optimal solution of the linear relaxation becomes infeasible.

Solution:

The optimal solution has $x_2 = 0$ and $x_4 = 0$, hence inserting the cut in the problem the optimal solution becomes infeasible.

The data in Python format:

```
from fractions import Fraction
import numpy as np
np.set_printoptions(precision=3,suppress=True)

c=np.array([4, -2])
A = np.array([[ 1, 2],
              [ 0, 3],
              [ 3, -3]])

b=np.array([2, 2, 2])
```