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DM545/DM871 – Linear and Integer Programming

Exercise 1

Consider the matrices:

$$A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

In each part compute the given expression. Where the computation is not possible explain why.

1. $D + E$
2. $D - E$
3. $5A$
4. $2B - C$
5. $2(D + 5E)$
6. $(C^T B)A^T$
7. $2\text{tr}(AB)$
8. $\det(E)$

Solution:

Taken by andrm17.

1) $D + E$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -2+0 & 1+3 & 8+0 \\ 3+-5 & 0+1 & 2+1 \\ 4+7 & -6+6 & 3+2 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 8 \\ -2 & 1 & 3 \\ 11 & 0 & 5 \end{bmatrix}$$

2) $D - E$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} - E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -2-0 & 1-3 & 8-0 \\ 3--5 & 0-1 & 2-1 \\ 4-7 & -6-6 & 3-2 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 8 \\ 8 & -1 & 1 \\ -3 & -12 & 1 \end{bmatrix}$$

3) $5A$

$$5 \cdot \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 & 5 \cdot 0 \\ 5 \cdot -4 & 5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -20 & 30 \end{bmatrix}$$

4) $2B - C$

$$2 \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot -7 & 2 \cdot 2 \\ 2 \cdot 5 & 2 \cdot 3 & 2 \cdot 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 4 \\ 10 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

We cannot perform a subtraction with the two matrices since they do not have the same dimensions.

5) $2(D + 5E)$

$$2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ n4 & -6 & 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} \right) = 2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 5 \cdot 0 & 5 \cdot 3 & 5 \cdot 0 \\ 5 \cdot -5 & 5 \cdot 1 & 5 \cdot 1 \\ 5 \cdot 7 & 5 \cdot 6 & 5 \cdot 2 \end{bmatrix} \right)$$

$$2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 15 & 0 \\ -25 & 5 & 5 \\ 35 & 30 & 10 \end{bmatrix} \right) = 2 \cdot \left(\begin{bmatrix} -2+0 & 1+15 & 8+0 \\ 3+-25 & 0+5 & 2+5 \\ 4+35 & -6+30 & 3+10 \end{bmatrix} \right)$$

$$2 \cdot \begin{bmatrix} -2 & 16 & 8 \\ -22 & 5 & 7 \\ 39 & 24 & 13 \end{bmatrix} = \begin{bmatrix} 2 \cdot -2 & 2 \cdot 16 & 2 \cdot 8 \\ 2 \cdot -22 & 2 \cdot 5 & 2 \cdot 7 \\ 2 \cdot 39 & 2 \cdot 24 & 2 \cdot 13 \end{bmatrix} = \begin{bmatrix} -4 & 32 & 16 \\ -44 & 10 & 14 \\ 78 & 48 & 26 \end{bmatrix}$$

6) $(C^T B)A^T$

$$\left(\begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}^T = \left(\begin{bmatrix} 4 & -3 & 2 \\ 9 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix}$$

We cannot perform this multiplication since C^T doesn't have the same number of columns as B has rows.

7) $2tr(AB)$

$$2tr \left(\begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \right) = 2tr \left(\begin{bmatrix} 2 \cdot 1 + 0 \cdot 5 & 2 \cdot -7 + 0 \cdot 3 & 2 \cdot 2 + 0 \cdot 0 \\ -4 \cdot 1 + 6 \cdot 5 & -4 \cdot -7 + 6 \cdot 3 & -4 \cdot 2 + 6 \cdot 0 \end{bmatrix} \right)$$

$$2tr \left(\begin{bmatrix} 2 & -14 & 4 \\ 26 & 46 & -8 \end{bmatrix} \right)$$

Trace is not defined for non-square matrices.

8) $det(E)$

We're using cofactor expansion to get the determinator.

$$det \left(\begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} \right) = 0 \cdot \begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix} - 3 \cdot \begin{bmatrix} -5 & 1 \\ 7 & 2 \end{bmatrix} - 0 \cdot \begin{bmatrix} -5 & 1 \\ 7 & 6 \end{bmatrix}$$

we can discard the zeroes and we're then left with:

$$-3 \cdot (-5 \cdot 2 - 7 \cdot 1) = -3 \cdot (-10 - 7) = -3 \cdot -17 = 51$$

Exercise 2

Consider the following system of linear equations in the variables $x, y, z \in \mathbb{R}$.

$$\begin{aligned} -2y + 3z &= 3 \\ 3x + 6y - 3z &= -2 \\ -3x - 8y + 6z &= 5 \end{aligned}$$

1. Write the augmented matrix of this system.
2. Reduce this matrix to row echelon form by performing a sequence of elementary row operations.
3. Solve the system and write its general solution in parametric form.

Solution:

```
import numpy as np

# The augmented matrix
AA = np.array([[ 0, -2, 3, 3],
             [ 3, 6, -3, -2],
             [-3, -8, 6, 5]])

In [30]: import sympy as sy
    ...: # np.linalg.solve(A,b)
    ...:
    ...: sy.Matrix(AA).rref()
    ...:
Out[30]:
(Matrix([
 [1, 0, 2, 7/3],
 [0, 1, -3/2, -3/2],
 [0, 0, 0, 0]]), (0, 1))
```

Hence, the solution is:

$$\mathbf{x} = \begin{bmatrix} 7/3 - 2t \\ -3/2 + 3/2t \\ t \end{bmatrix} = \begin{bmatrix} 7/3 \\ -3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 3/2 \\ 1 \end{bmatrix} t \qquad t \in \mathbb{R}$$

Exercise 3

Consider the following matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

1. Find M^{-1} by performing row operations on the matrix $[M \mid I]$.
2. Is it possible to express M as a product of elementary matrices? Explain why or why not.

Solution:

```
import numpy as np

M = np.array([[ 1, 0, 1],[-1,1,0],[2,2,2]])
MM = np.concatenate([M,np.identity(3)],axis=1)

import sympy as sy
sy.Matrix(MM).rref()
```

```
(Matrix([
 [1, 0, 0, -1.0, -1.0,  0.5],
 [0, 1, 0, -1.0,    0,  0.5],
 [0, 0, 1,  2.0,  1.0, -0.5]]), (0, 1, 2))
```

Yes, it is possible. Since the matrix is invertible we have shown above that we can go from an identity matrix to M^{-1} and consequently also to M from an identity matrix with elementary row operations. Elementary row operations can be expressed as products between elementary matrices.

Exercise 4

1. Given the point $[3, 2]$ and the vector $[-1, 0]$ find the vector and parametric (Cartesian) equation of the line containing the point and parallel to the vector.
2. Find the vector and parametric (Cartesian) equations of the plane in $\mathbb{R}^3$ that passes through the origin and is orthogonal to $\mathbf{v} = [3, -1, -6]$.

Solution:

The vector equation: Let $[3, 2]^T = \mathbf{p}$ and $[-1, 0]^T = \mathbf{v}$. Any point $\mathbf{x}$ on the line can be expressed as:

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \qquad \forall t \in \mathbb{R}$$

We can derive the Cartesian equation by eliminating t from the equation above:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

From the first coordinate: $x_1 = 3 - t$ and from the second: $x_2 = 2$. The Cartesian equation is $x_2 = 2$ since x_1 is free to get any value.

The plane through the origin orthogonal to $\mathbf{v} = [3, -1, -6]$ is given by:

$$\mathbf{x}^T\mathbf{v} = 0$$

That is: $3x_1 - x_2 - 6x_3 = 0$.