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DM545/DM871 – Linear and integer programming

Sheet 3, Spring 2020 [pdf format]

Starred exercises are relevant for the tests.

Solution:

Included. The HTML may not be well formatted. See PDF version.

Exercise 1*

Show that the dual of $\max\{c^T x | Ax = b, x \geq 0\}$ is $\min\{y^T b | y^T A \geq c\}$. Use one of the methods presented in class or even all of them.

Solution:

Let's show it here by the bounding method.
Given $\max\{c^T x \mid Ax = b, x \geq 0\}$ we search for multipliers $y \in \mathbb{R}^n$ such that $y^T Ax = y^T b$ (since we have equalities, the multipliers can be both positive or negative as we do not need to ensure the maintainance of the direction of the inequality). To ensure that we find an upper bound and hence have $c^T x \leq y^T Ax$, we impose $y^T A \geq c^T$ (since $x \geq 0$). Hence, the best upper bound will be given by solving $\min\{y^T b \mid y^T A \geq c^T\}$ (recalling from linear algebra that $(AB)^T = B^T A^T$, we can rewrite: $\min\{y^T b \mid A^T y \geq c\}$, which is the form we would obtain using the recipe method.)

Exercise 2*

Consider the following LP problem:

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 30 \\ & x_1 + 2x_2 \geq 10 \\ & x_1 - x_2 \leq 1 \\ & x_2 - x_1 \leq 1 \\ & x_1 \geq 0 \end{aligned}$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Solution:

The dual is:

$$\begin{aligned} \max\ & 30y_1 + 10y_2 + y_3 + y_4 \\ & 2y_1 y_2 + y_3 - y_4 \geq 2 \\ & 3y_1 + 2y_2 - y_3 + y_4 = 3 \\ & y_1, y_3, y_4 \geq 0 \\ & y_2 \leq 0 \end{aligned}$$

We use the complementary slackness theorem.

$$\begin{cases} 2y_1 y_2 + y_3 - y_4 = 2 \\ 3y_1 + 2y_2 - y_3 + y_4 = 3 \\ y_2 = 0 \\ y_3 = 0 \end{cases}$$

The first because the corresponding variable of the primal is > 0, the second for the same reason or however because it is already tight by definition, the third and fourth equation are a consequence of the fact that substituting the value of the primal variables variables in the primal problem, the second and third constraints are binding. What we obtain is a linear system of four equations in four variables that we can solve to find the value of the variables of the dual problem.

Exercise 3*

Consider the problem

$$\begin{array}{ll} \text{maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]

Exercise 4*

Consider the following LP:

$$\begin{array}{ll} \min & 3x_1 + 2x_2 - 4x_3 \\ & 2x_1 + x_2 + x_3 \geq 3 \\ & x_1 + x_2 + 2x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Find the optimal solution knowing that the solution of the dual problem is $(u_1, u_2) = (10/3, 11/3)$.

Exercise 5*

An investor has 10,000 Dkk to invest in four projects. The following table gives the cash flow for the four investments.

Project	Year 1	Year 2	Year 3	Year 4	Year 5
1	-1.00	0.50	0.30	1.80	1.20
2	-1.00	0.60	0.20	1.50	1.30
3	0.00	-1.00	0.80	1.90	0.80
4	-1.00	0.40	0.60	1.80	0.95

The information in the table can be interpreted as follows: For project 1, 1.00 Dkk invested at the start of year 1 will yield 0.50 Dkk at the start of year 2, 0.30 Dkk at the start of year 3, 1.80 Dkk at the start of year 4, and 1.20 Dkk at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. The investor has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of 1 year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.
[Taken from Operations Research: An Introduction, Taha]

Solution:

Let
x_i = Krone invested in project $i, i = 1, 2, 3, 4$
y_j = Krone invested in bank in year $j, j = 1, 2, 3, 4$

$$\begin{aligned} \max z = & y_5 \\ & x_1 + x_2 + x_4 + y_1 \leq 10000 \\ & .5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_1 - y_2 = 0 \\ & .3x_1 + .2x_2 + .8x_3 + .6x_4 + 1.065y_2 - y_3 = 0 \\ & 1.8x_1 + 1.5x_2 + 1.9x_3 + 1.8x_4 + 1.065y_3 - y_4 = 0 \\ & 1.2x_1 + 1.3x_2 + .8x_3 + .95x_4 + 1.065y_4 - y_5 = 0 \\ & x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

Expected Investment Cash Flows and Net Present Value	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6	Budget
Year 1	-$5.00	-$9.00	-$12.00	-$7.00	-$20.00	-$18.00	$45.00
Year 2	-$6.00	-$6.00	-$10.00	-$5.00	$6.00	-$15.00	$30.00
Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00	$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00	$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00	$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00	$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93	

Figure 1:

Optimum solution:
$x_1 = 0, x_2 = 10,000, x_3 = 6000, x_4 = 0$
$y_1 = 0, y_2 = 0, y_3 = 6800, y_4 = 33,642$
$z = 53,628.73$ at the start of year 5

Exercise 6* Budget Allocation

A company has six different opportunities to invest money. Each opportunity requires a certain investment over a period of 6 years or less. See Figure 1.
The company wants to invest in those opportunities that maximize the combined *Net Present Value* (NPV). It also has an investment budget that needs to be met for each year. (The Net Present Value is calculated with an interest rate of 5%).
How should the company invest?
We assume that it is possible to invest partially in an opportunity. For instance, if the company decides to invest 50% of the required amount in an opportunity, the return will also be 50%.

Net present value:
A debtor wants to delay the payment back of a loan for t years. Let P be the value of the original payment presently due. Let r be the market rate of return on a similar investment asset. The future value of P is

$$F = P(1 + r)^t$$

Viceversa, consider the task of finding the present value P of $100 that will be received in five years, or equivalently, which amount of money today will grow to $100 in five years when subject to a constant discount rate. Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1+r)^t} = \frac{\$100}{(1+0.05)^5} = \$78.35.$$

Solution:

Net Present Value calculation:
for each opportunity we calculate the NPV at time zero (the time of decision) as:

$$P_0 = \sum_{t=1}^{5} \frac{F_t}{(1+0.05)^5}$$

Let B_t be the budget available for investments during the years $t = 1..5$. Let a_{tj} be the cash flow for opportunity j and c_j its NPV. We want to choose a set of opportunities such that the budget is never exceeded and the expected return is maximized. We consider divisible opportunities.

Variables $x_j = 1$ if opportunity j is selected and $x_j = 0$ otherwise, $j = 1..6$

Objective

$$\max \sum_{j=1}^{6} c_j x_j$$

Constraints

$$\textstyle\sum_{j=1}^{6} a_{tj} x_j + B_t \geq 0 \qquad \forall t = 1..5$$