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DM545/DM871 – Linear and integer programming

Sheet 4, Spring 2020 [pdf format]

Starred exercises are relevant for the tests.

Solution:
Included. The HTML may not be well formatted. See PDF version.

Exercise 1*

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- Solving an an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. The solution of the auxiliary LP problem gives an initial feasible basis or a proof of infeasibility.
- Applying the dual simplex on a possibly modified problem to find a feasible solution. If the initial infeasible tableau of the original problem is not optimal then the objective function can be temporarily modified for this phase in order to make the initial tableau optimal although not feasible. Opposite to the primal simplex method, the dual simplex method iterates through infeasible basis solutions, while maintaining them optimal, and stops when a feasible solution is reached.

Dual Simplex: The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:

1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| \; : \; a_{i^*j} < 0\}$
3. Update the tableau around the pivot in the same way as with the primal simplex.
4. Stop if no right-hand side term is negative.

Duality can help us with the issue of initial feasible basis solutions. In the problem above, if the objective function was $w = -x_1 - x_2$, then the initial basis solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. But with objective function z the simplex has infeasible initial basis in both problems. However we can change temporarily the objective function z with w and apply the dual simplex method. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and

continue iterating with the primal simplex. The phase I–phase II simplex method that uses the dual simplex is also called the *dual-primal simplex method*.
Apply this method to the problem above and verify that it leads to the same solution as in point 1.

Solution:

$$\begin{array}{rrcrcl} \max & x_1 & - & x_2 & = & z \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

We put in equational standard form by introducing a slack variable $s_1 \geq 0$ and a surplus variable $s_2 \geq 0$:

$$\begin{array}{rrcrcrcrcl} \max & x_1 & - & x_2 & = & z & & & & \\ & x_1 & + & x_2 & + & s_1 & & & = & 2 \\ & 2x_1 & + & 2x_2 & & & - & s_2 & = & 2 \\ & & & & & & x_1, x_2, & s_1, s_2 & \geq & 0 \end{array}$$

This form is not canonical and therefore the first tableau does not have a feasible starting solution.

Auxiliary Problem Approach

We proceed by

- Phase I solving an auxiliary/augmented problem
- Phase II continuing with ordinary simplex

Phase I We introduce an auxiliary variable $a_1 \geq 0$ in the constraint that makes the infeasibility to yield a canonical form:

$$\begin{array}{rrcrcrcrcrcl} \max & x_1 & - & x_2 & = & z & & & & & & \\ & x_1 & + & x_2 & + & s_1 & & & & & = & 2 \\ & 2x_1 & + & 2x_2 & & & - & s_2 & + & a_1 & = & 2 \\ & & & & & & & x_1, x_2, s_1, & s_2, & a_1 & \geq & 0 \end{array}$$

Now we have a canonical form

```
| x1 | x2 | s1 | s2 | a1 | -z | b |
|----+----+----+----+----+----+---|
|  1 |  1 |  1 |  0 |  0 |  0 | 2 |
|  2 |  2 |  0 | -1 |  1 |  0 | 2 |
|  1 | -1 |  0 |  0 |  0 |  1 | 0 |
|----+----+----+----+----+----+---|
```

This problem will have the same solution as the original one only when $a_1 = 0$. We can then solve

- an *augmented problem* by introducing the following objective function $\max w = x_1 - x_2 - Ma_1$, where M is a large enough constant or
- an *auxiliary* problem $\min w = a_1 = -\max(-a_1)$.

Let's take the auxiliary problem, if $w^* > 0$ then we will conclude that the feasibility region of the orginal problem is empty. Otherwise, if $w^* = 0$, then this implies that $a_1 = 0$ and we found a feasible solution. Let's proceed by setting up the tableau of the auxiliary problem

```
| x1 | x2 | s1 | s2 | a1 | -z | -w | b |
|----+----+----+----+----+----+----+---|
|  1 |  1 |  1 |  0 |  0 |  0 |  0 | 2 |
```

```
| 2 |  2 | 0 | -1 |  1 | 0 | 0 | 2 |
| 1 | -1 | 0 |  0 |  0 | 1 | 0 | 0 |
| 0 |  0 | 0 |  0 | -1 | 0 | 1 | 0 |
|----+----+----+----+----+----+----+---|
```

This is not in canonical form but it is easy to bring it to canonical form: just add the second row to the last one.

```
| x1 | x2 | s1 | s2 | a1 | -z | -w | b |
|----+----+----+----+----+----+----+---|
|  1 |  1 |  1 |  0 |  0 |  0 |  0 | 2 |
|  2 |  2 |  0 | -1 |  1 |  0 |  0 | 2 |
|  1 | -1 |  0 |  0 |  0 |  1 |  0 | 0 |
|  2 |  2 |  0 | -1 |  0 |  0 |  1 | 2 |
|----+----+----+----+----+----+----+---|
```

The variables s_1, a_1 give us a feasible basis now. It is not optimal. We proceed with the pivot operations. In this case it is worth noting that in the ratio rule, we do not consider the third row since that row corresponds to the orginal objective function and not to a constraint.
We make x_1 enter the basis and consequently a_1 goes out. The pivot is 2 and the new tableau:

```
|             | x1 | x2 | s1 | s2   | a1   | -z | -w |  b |
|------------+----+----+----+------+------+----+----+----|
| R1'=R1-R2' |  0 |  0 |  1 | 1/2  | -1/2 |  0 |  0 |  1 |
| R2'=R2/2   |  1 |  1 |  0 | -1/2 | 1/2  |  0 |  0 |  1 |
| R3'=R3-R2' |  0 | -2 |  0 | 1/2  | -1/2 |  1 |  0 | -1 |
| R4'=R4-R2  |  0 |  0 |  0 | 0    | -1   |  0 |  1 |  0 |
|------------+----+----+----+------+------+----+----+----|
```

The tableau is optimal. One non basic variable has reduced cost null, which indicates that there are infinite solutions, but this is not relevant now. The relevant thing is that $w^* = 0$ hence the minimum of the auxiliary problem is 0 and hence there is a feasible solution for $a_1 = 0$. This concludes the Phase I of the algorithm since a feasible solution for the auxiliary problem is feasible also for the original problem.

Phase II We throw away the last row and the second last column from the tableau since we do not need them anymore.

```
| x1 | x2 | s1 | s2   | a1   | -z |  b |
+----+----+----+------+------+----+----|
|  0 |  0 |  1 | 1/2  | -1/2 |  0 |  1 |
|  1 |  1 |  0 | -1/2 | 1/2  |  0 |  1 |
|  0 | -2 |  0 | 1/2  | -1/2 |  1 | -1 |
+----+----+----+------+------+----+----|
```

The tableau is not optimal. The basic solution corresponding to this tableau is feasible but not optimal. We bring s_2 in the basis and make s_1 leave. The new tableau is:

```
|            | x1 | x2 | s1 | s2 |  a1 | -z |  b |
```

```
|-----------+----+----+----+----+-----+----+----|
| R1'=2*R1  |  0 |  0 |  2 |  1 |  -1 |  0 |  2 |
| R2'=R2+R1 |  1 |  1 |  0 |  0 | 1/2 |  0 |  2 |
| R3'=R3-R1 |  0 | -2 | -1 |  0 |   0 |  1 | -2 |
|-----------+----+----+----+----+-----+----+----|
```

The tableau is now optimal. The optimal solution is $\mathbf{x} = (2, 0)$ and $z^* = 2$.

Dual-Primal Simplex Method

Phase I Let's write the dual of the problem above:

$$\begin{array}{rcrcl} \max & x_1 & - & x_2 & = z \\ & x_1 & + & x_2 & \leq 2 \\ & 2x_1 & + & 2x_2 & \geq 2 \\ & & & x_1, x_2 & \geq 0 \end{array} \qquad \begin{array}{rcrcl} \min & 2y_1' & + & 2y_2' & = w \\ & y_1' & + & 2y_2' & \geq 1 \\ & y_1' & + & 2y_2' & \geq -1 \\ & & & y_1' & \geq 0 \\ & & & y_2' & \leq 0 \end{array} \quad \xrightarrow[y_2'=-y_2]{y_1'=y_1}$$

$$\begin{array}{rcrcl} \min & 2y_1 & - & 2y_2 & = w \\ & y_1 & - & 2y_2 & \geq 1 \\ & y_1 & - & 2y_2 & \geq -1 \\ & & & y_1, y_2 & \geq 0 \end{array}$$

If we put this LP problem in standard form:

$$\begin{array}{rcrcl} \max & -2y_1 & + & 2y_2 & = w \\ & -y_1 & + & 2y_2 & \leq -1 \\ & -y_1 & + & 2y_2 & \leq 1 \\ & & & y_1, y_2 & \geq 0 \end{array}$$

and looking at the tableau:

```
| y1 | y2 | s1 | s2 | -z |  b |
|----+----+----+----+----+----|
| -1 |  2 |  1 |  0 |  0 | -1 |
| -1 |  2 |  0 |  1 |  0 |  1 |
| -2 |  2 |  0 |  0 |  1 |  0 |
|----+----+----+----+----+----|
```

we see that the initial tableau like for the primal problem is infeasible.
However, the dual problem has an advantage, if we change temporarily the objective function of the primal problem to $\eta = -x_1 - x_2$, the dual problm becomes:

$$\begin{array}{r} \max -x_1 - x_2 = \eta \\ x_1 + x_2 \leq 2 \\ 2x_1 + 2x_2 \geq 2 \\ x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{r} \min 2y_1 - 2y_2 = \gamma \\ y_1 - 2y_2 \geq 1 \\ y_1 - 2y_2 \geq -1 \\ y_1, y_2 \geq 0 \end{array} \qquad \begin{array}{r} \max -2y_1 + 2y_2 = \gamma \\ -y_1 + 2y_2 \leq 1 \\ -y_1 + 2y_2 \leq 1 \\ y_1, y_2 \geq 0 \end{array}$$

and the corresponding tableau has an easy basic feasible solution:

```
| y1 | y2 | s1 | s2 | -z | b |
|----+----+----+----+----+---|
| -1 |  2 |  1 |  0 |  0 | 1 |
| -1 |  2 |  0 |  1 |  0 | 1 |
| -2 |  2 |  0 |  0 |  1 | 0 |
|----+----+----+----+----+---|
```

We can then solve to optimality with the primal simplex: the variable y_2 enters the basis and the variable s_2 exits. The new tableau becomes:

	y1	y2	s1	s2	-z	b
R1	-1	2	1	0	0	1
R2'=R2/2	-1/2	1	0	1/2	0	1/2
R3'=R3-R2	-1	0	0	-1	1	-1/2

and it is optimal. At this stage we can go back to the primal problem where we now have a feasible solution, change the objective function back to the original one and continue with the primal simplex.

We can do the same iteration on the primal but with the dual simplex. Let's write the tableau of the primal with the objective function temporarily changed and keeping the old objective as well:

x1	x2	s1	s2	-z	-e	b
1	1	1	0	0	0	2
-2	-2	0	1	0	0	-2
1	-1	0	0	1	0	0
-1	-1	0	0	0	1	0

As we see we have the conditions of the dual simplex satisfied, the tableau is optimal but not feasible. Let's make an iteration of the dual simplex. We choose the row with negative b term and the column with negative pivot that minimizes the ratio test: $|c/a|$. We choose the second row and the second column (again watch out that we do not consider the row of the addd old objective to decide the row). In other terms we try to make the solution feasible while minimizing the loss in quality. The opertations to update the tableau remain the same as for the primal simplex. We obtan:

	x1	x2	s1	s2	-z	-e	b
R1'=R1-R2'	0	0	1	1/2	0	0	1
R2'=-1/2R2	1	1	0	-1/2	0	0	1
R3'=R3+R2'	2	0	0	-1/2	1	0	1
R4'=R4+R2'	0	0	0	-1/2	0	1	1

This tableau is optimal for the dual simplex, this means that a feasible solution for the primal problem has been found: $(0, 1, 1, 0)$. We can now proceed with the primal simplex.
Note that the considerations on the dual problem made above were just for explanation purposes, when solving our LP problem we do not need to write down the dual form of it or its tableaux. Instead, we just need to switch from dual simplex to primal simplex always working on the original (the primal) formulation of the problem. The dual simplex method simply a new way of picking the entering and leaving variables in a sequence of primal tableaux.

Phase II We can now remove the temporary objective function and the corresponding column and proceed with the primal simplex.

```
| x1 | x2 | s1 | s2   | -z | b |
|----+----+----+------+----+---|
|  0 |  0 |  1 | 1/2  |  0 | 1 |
|  1 |  1 |  0 | -1/2 |  0 | 1 |
|  2 |  0 |  0 | -1/2 |  1 | 1 |
|----+----+----+------+----+---|
```

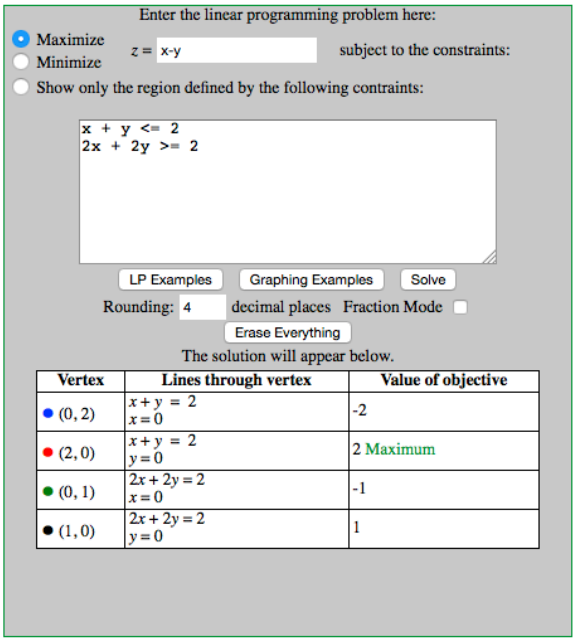
x_1 enters the basis and x_2 exits. The tableau is updated consequently:

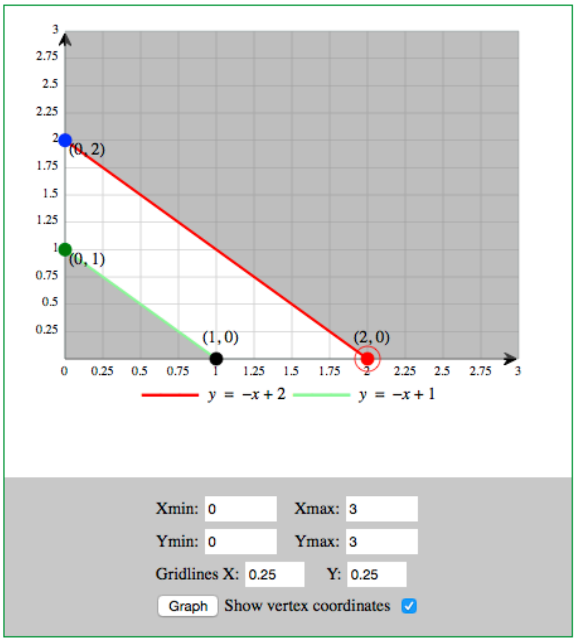
```
|              | x1 | x2 | s1 | s2   | -z |  b |
|--------------+----+----+----+------+----+----|
| R1'=R1       |  0 |  0 |  1 | 1/2  |  0 |  1 |
| R2'=R2       |  1 |  1 |  0 | -1/2 |  0 |  1 |
| R3'=R3-2*R2  |  0 | -2 |  0 | 1/2  |  1 | -1 |
|--------------+----+----+----+------+----+----|
```

A reduced cost is still positive, hence we make s_2 enters in the basis and s_1 leave. This leads to

```
|            | x1 | x2 | s1 | s2 | -z |  b |
|------------+----+----+----+----+----+----|
| R1'=2*R1   |  0 |  0 |  2 |  1 |  0 |  2 |
| R2'=R2+R1  |  1 |  1 |  1 |  0 |  0 |  2 |
| R3'=R3-R1  |  0 | -2 | -1 |  0 |  1 | -2 |
|------------+----+----+----+----+----+----|
```

The tableau is now optimal and the corresponding basic feasible solution is $\mathbf{x} = (2, 0)$ and has value $z^* = 2$.
We can visualize the problem using the LP Grapher tool linked from the course webpage:





Exercise 2* Sensitivity Analysis and Revised Simplex

A furniture-manufacturing company can produce four types of product using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.
- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.
- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve the following LP problem

$$\begin{aligned} \max \quad & 19x_1 + 13x_2 + 12x_3 + 17x_4 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 \\ & x_1 + x_2 + x_3 + x_4 \leq 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The final tableau has x_1, x_3 and x_4 in basis. With the help of a computational environment such as Python for carrying out linear algebra operations, address the following points:

a) Write A_B, A_N, $A_B^{-1}A_N$, the final simplex tableau and verify that the solution is indeed optimal.

Solution:

The initial tableau is:

```
|-----+-----+-----+-----+-----+-----+-----+----+-----|
| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | -z |   b |
|-----+-----+-----+-----+-----+-----+-----+----+-----|
|   3 |   2 |   1 |   2 |   1 |   0 |   0 |  0 | 225 |
|   1 |   1 |   1 |   1 |   0 |   1 |   0 |  0 | 117 |
|   4 |   3 |   3 |   4 |   0 |   0 |   1 |  0 | 420 |
|  19 |  13 |  12 |  17 |   0 |   0 |   0 |  1 |   0 |
|-----+-----+-----+-----+-----+-----+-----+----+-----|
```

We know that there will be 3 variables in basis. The text of the problem tells us which these 3 variables are: 1, 3, 4. Hence,

$$A_B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix} \qquad A_N = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

We can calculate $A_B^{-1}A_N$ in Python or in R:

```
> B=matrix(c(3,1,2,1,1,1,4,3,4),byrow=TRUE,ncol=3)
> B1=solve(B)
> B%*%B1  # check to make sure it is correct!
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
> N=matrix(c( 2, 1, 0, 0, 1, 0, 1, 0, 3, 0, 0, 1),ncol=4,byrow=TRUE)
> B1%*%N
     [,1] [,2] [,3] [,4]
[1,]    1    1    2   -1
[2,]    1    0    4   -1
```

```
[3,]    -1    -1    -5     2
> cN=c(13,0,0,0)
> cB=c(19,12,17)
> cN-cB%*%B1%*%N
      [,1] [,2] [,3] [,4]
[1,]    -1    -2    -1    -3
> cB%*%B1%*%c(225,117,420)
> 1827
```

This code gives us:

$$\bar{A} = A_B^{-1} A_N = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 4 & -1 \\ -1 & -1 & -5 & 2 \end{bmatrix} \qquad x_B^* = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix}$$

$$\bar{c}_N = \begin{bmatrix} \bar{c}_2 & \bar{c}_5 & \bar{c}_6 & \bar{c}_7 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 & -3 \end{bmatrix}$$

and we can write the final tableau as:

```
|-----+-----+-----+-----+-----+-----+-----+----+-------|
| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | -z |     b |
|-----+-----+-----+-----+-----+-----+-----+----+-------|
|   1 |   1 |   0 |   0 |   1 |   2 |  -1 |  0 |    39 |
|   0 |   1 |   1 |   0 |   0 |   4 |  -1 |  0 |    48 |
|   0 |  -1 |   0 |   1 |  -1 |  -5 |   2 |  0 |    30 |
|   0 |  -1 |   0 |   0 |  -2 |  -1 |  -3 |  1 | -1827 |
|-----+-----+-----+-----+-----+-----+-----+----+-------|
```

Since all reduced costs are negative then the tableau and the corresponding solution are optimal.

b) What is the increase in price (reduced cost) that would make product x_2 worth to be produced?

Solution:

The increase in price of a quantity strictly larger than 1 would make the product 2 worth being produced. Indeed, let $c_2' = c_2 + \delta$ be the new price. We know that the coefficient in the objective function goes in the reduced cost calculation multiplied by 1. Hence, to have a positive reduced cost we have:

$$-1 + \delta > 0 \qquad \Longrightarrow \qquad \delta > 1$$

We could also recalcuate the reduced cost from scratch using the multipliers π: $c_2' + \sum_{i=1}^{3} \pi_i a_{i2}$. The value of π_i are read from the final tableau and they correspond to the reduced costs of the slack variables, ie, $(-2, -1, -3)$.

c) What is the marginal value (shadow price) of an extra hour of work or amount of metal and wood?

Solution:

The marginal values are the values of the dual variable y_1, y_2, y_3. From the strong duality theorem, we know that $y_i = -\pi_i = -\bar{c}_{n+i}$, $i = 1..m$. Hence, $\mathbf{y} = (2, 1, 3)$.

An extra hour of work has marginal value of 2, that is, having one unit more of work would improve the revenue by 2. For the other two resources the marginal values are 1 and 3, respectively.

We can cross check these conclusions: by the complementary slackness theorem, the fact that all three dual variables are strictly positive indicates that all three constraints in the primal are active≡tight≡ binding. Hence, it makes sense to have that an increase in the capacity of those constraints implies an increase in the profit. The conclusion that all three constraints are tight can be also reached by the fact that the slack variables are 0 in the final tableau. If some constraint was not tight, then the marginal value of the corresponding resource would be zero since an increase in its capacity does not imply an immediate improvement in total profit.

d) Are all resources totally utilized, i.e. are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

Solution:

Since all dual variables are strictly larger than zero, then all constraints are binding. Indeed for the complementary slackness theorem, we have that:

$$\left(b_i - \sum_{j=1}^{n} a_{ij}x_j^*\right) y_i^* = 0, \quad i = 1, \dots, m$$

e) From the economical interpretation of the dual why product x_2 is not worth producing? What is its imputed cost?

Solution:

It is not worth producing 2 because $\sum_i y_i a_{i2} > c_2$, that is, we are better off selling the raw materials to produce the product. Indeed y_i is the price of one unit of resource i and a_{i2} is the amount of i necessary to produce 2.

$$\sum_i y_i a_{i2} = 2 * (2) + 1 * (1) + 3 * (3) = 14 > 13$$

Solve the following variations:

1. The net profit brought in by each desk increases from 13 Euro to 15 Euro.

Solution:

We saw earlier that if the price of product 2 increases by more than 1 then the reduced cost becomes positive and it enters the basis. We can iterate the revised simplex as follows:

Step 1 and 2 to determine the entering varible are already done in the point a) above.

We need to do Step 3 to determine the leaving variable: we need to find the constraint that limit the increase of x_2, *theta*. We solve first $A_B d = a$ in d. Here, $\mathbf{a}$ is the column of the matrix A (augmented with the slack variables) from the initial tableau corresponding to the entering variable x_2. We use the inverse of A_B calculated earlier in a) above in R:

```
> B1%*%c(2,1,3)
     [,1]
[1,]    1
[2,]    1
[3,]   -1
```

that is

$$\mathbf{d} = A_B^{-1}\mathbf{a} = A_B^{-1}\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Then the new solutioon x_B is derived from the old one by means of $\mathbf{d}$ and the increase θ:

$$x_B = \begin{bmatrix}x_1\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}39\\48\\30\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix}\theta \geq 0$$

The increase θ must be such that the value of the variables still remains feasible, ie, $x_i \geq 0$. Hence $\theta \leq 39$ and the leaving variable is x_1, since it is the one that goes to zero. The new solutions is

$$x_B = \begin{bmatrix}x_2\\x_3\\x_4\end{bmatrix} == \begin{bmatrix}39\\48\\30\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix}\theta = \begin{bmatrix}39-39\\48-39\\30+39\end{bmatrix} = \begin{bmatrix}0\\9\\69\end{bmatrix}$$

and the objective value:

```
> c=c(19,15,12,17)
> c%*%c(0,39,9,69)
     [,1]
[1,] 1866
```

2. The availability of metal increases from 117 to 125 units per day

   Solution:

   This is a change in the RHS term of constraint 2. The optimality of the current solution does not change, since all reduced costs stay negative, but we need to check if we are still feasible. We need to look at the final tablea and recompute the b of all constraints. We can do this with $A_B^{-1}b$:

```
> b=c(225,125,420)
> B1%*%b
     [,1]
[1,]   55
[2,]   80
[3,]  -10
```

   The last cosntraint becomes negative, hence we need to iterate with the dual simplex.

3. The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood and bring in a net profit of 14 Euro.

   Solution:

   We need to check if the reduced cost of the new variable would become positive by computing $c_0 + \sum_i \pi_i a_{ij}$:

```
> 14-3*2-1*1-2*3
[1] 1
```

   which is positive, hence we need to iterate as done in point 1).

4. The number of chairs produced must be at most five times the numbers of desks

   Solution:

   This corresponds to introduce a new constraint: $x_3 \leq 5x_2$. In the new standard form we have a new slack variable x_8. Adding the constraint in the tableau and bringing back the tableau in canonical standard form we observe that a RHS term becomes negative. Hence, we need to iterate with the dual simplex. After on iteration with the dual simplex, the final tableau becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 4/3 & -5/6 & 1/6 & 0 & 31 \\ 0 & 1 & 0 & 0 & 0 & 2/3 & -1/6 & -1/6 & 0 & 8 \\ 0 & 0 & 0 & 1 & -1 & -13/3 & 11/6 & -1/6 & 0 & 38 \\ 0 & 0 & 1 & 0 & 0 & 10/3 & -5/6 & 1/6 & 0 & 40 \\ 0 & 0 & 0 & 0 & -2 & -1/3 & -19/6 & -1/6 & 1 & -1819 \end{bmatrix}$$

   If after the introduction of the constraint the current solution had stayed feasible then we would have needed to check whether its was also optimal. We can either repeat the steps done at part 1 above to compute the new reduced costs or we can include the new row in the final tableau and proceed to put the tableau in canonical form. Then we look at the value of the reduced costs.

Exercise 3

Solve the systems $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1\ 2\ 3]$ and $E_1 E_2 E_3 E_4 \mathbf{d} = [1\ 2\ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution:

This exercise is to show that the two systems can be solved quite easily. Let's take first $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1\ 2\ 3]$, we use the backward transformation and solve the sequence of linear systems:

$$\mathbf{u}^T E_4 = [1\ 2\ 3], \quad \mathbf{v}^T E_3 = \mathbf{u}^T, \quad \mathbf{w}^T E_2 = \mathbf{v}^T, \quad \mathbf{y}^T E_1 = \mathbf{w}^T$$

$$\mathbf{u}^T \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [1, 2, 3]$$

Since the eta matrices have always one 1 in two columns then the solution can be read up easily. From the third column we find $u_3 = 3$. From the second column, we find $u_2 = 2$. Substituting in the first column, we find $-0.5u_1 + 3 * 2 + 1 * 3 = 1$, which yields $u_1 = 18$. The next syestem is:

$$\mathbf{v}^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [18, 2, 3]$$

From the first column we get $v_1 = 18$, from the second column $v_2 = 2$ from the last column $v_3 = 3/24$. The next:

$$\mathbf{w} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = [18, 2, 3/24]$$

...

Exercise 4*

Write the dual of the following problem

$$\begin{aligned}
(P) \qquad & \max \sum_{j \in J} \sum_{i \in I} r_j x_{ij} \\
& \sum_{j \in J} x_{ij} \leq b_i && \forall i \in I \\
& \sum_{i \in I} x_{ij} \leq d_j && \forall j \in J \\
& \sum_{i \in I} p_i x_{ij} = p_j \sum_{i \in I} x_{ij} && \forall j \in J \\
& x_{ij} \geq 0 && \forall i \in I, j \in J
\end{aligned}$$

Solution:

There are three different sets of constraints. We introduce the dual variables $\alpha_i \geq 0$, for $i \in I$, for the first set; the dual variables $\beta_j \geq 0$, for $j \in J$, for the second set; and the dual variables $\gamma_j \in \mathbb{R}$, for $j \in J$ for the third set.

We then write the A matrix for the example in the picture, augmented with the b vector:

and finally the dual from the columns of the A matrix in general terms:

$$\begin{aligned}
(D) \qquad & \min \sum_{i \in I} \alpha_i b_i + \sum_{j \in J} \beta_j d_j \\
& \alpha_i + \beta_j + (p_i - p_j)\gamma_j \geq r_j && \forall i \in I, j \in J \\
& \alpha_i \geq 0 && \forall i \in I \\
& \beta_j \geq 0 && \forall j \in J \\
& \gamma_j \in \mathbb{R} && \forall j \in J
\end{aligned}$$

x_{G1}	x_{G2}	x_{G3}	x_{G4}	x_{T1}	x_{T2}	x_{T3}	x_{T4}	
1	1	1	1	0	0	0	0	b_1
0	0	0	0	1	1	1	1	b_2
1				1				d_1
	1				1			d_2
		1				1		d_3
			1				1	d_4
$p_G - p1$				$p_T - p1$				
	$p_G - p2$				$p_T - p2$			
		$p_G - p3$				$p_T - p3$		
			$p_G - p4$				$p_T - p4$	
r_1	r_2	r_3	r_4	r_1	r_2	r_3	r_4	

Exercise 5 Factory Planning and Machine Maintenance

A firm makes seven products $1, \dots, 7$ on the following machines: 4 grinders, 2 vertical drills, 3 horizontal drills, 1 borer, and 1 planer.
Each product yields a certain contribution to the profit (defined as selling price minus cost of raw materials expressed in Euro/unit). These quantities (in Euro/unit) together with the production times (hours/unit) required on each process are given below.

product	1	2	3	4	5	6	7
profit	10	6	8	4	11	9	3
grinding	0.5	0.7	0	0	0.3	0.2	0.5
vdrill	0.1	0.2	0	0.3	0	0.6	0
hdrill	0.2	0	0.8	0	0	0	0.6
boring	0.05	0.03	0	0.07	0.1	0	0.08
planning	0	0	0.01	0	0.05	0	0.05

In the first month (January) and the five subsequent months certain machines will be down for maintenance. These machines will be:

January	1 grinder
February	2 hdrill
March	1 borer
April	1 vdrill
May	1 grinder
May	1 vdrill
June	1 planer
June	1 hdrill

There are marketing limitations on each product in each month. That is, in each month the amount sold for each product cannot exceed these values:

product	1	2	3	4	5	6	7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

It is possible to store products in a warehouse. The capacity of the storage is 100 units per product type per month. The cost is 0.5 Euro per unit of product per months. There are no stocks in the first month but it is desired to have a stock of 50 of each product type at the end of June.
The factory works 6 days a week with two shifts of 8 hours each day. (It can be assumed that each month consists of 24 working days.)
The factory wants to determine a production plan, that is, the quantity to produce, sell and store in each month for each product, that maximizes the total profit.

Task 1 Model the factory planning problem for the month of January as an LP problem.

Solution:
The problem is taken from the book [Wi].
The problem is also one of Gurobi Examples:
http://www.gurobi.com/resources/examples/factory-planning-I
There is also a video: https://youtu.be/vnLc_3VnVcw?t=32m51s
You find the solutions also in this document.

Solution:
The objective is to find the optimum "product mix" subject to the production capacity and the marketing limitations. If storage of single products is not allowed, the model for January can be formulated as follows. Let the real variables x_i represent the quantities of product i to be made. Let GR, VD, HD, BR and PL stand for, respectively, grinding, vertical drilling, horizontal drilling, boring and planing. Let the total working hours for each machine be $8 * 2 * 24 = 384$.

$$
\begin{array}{lrcrcrcrcrcrcrl}
\max & 10x_1 & + & 6x_2 & + & 8x_2 & + & 4x_4 & + & 11x_5 & + & 9x_6 & + & 3x_7 & \\
GR: & 0.5x_1 & + & 0.7x_2 & + & & + & & + & 0.3x_5 & + & 0.2x_6 & + & 0.5x_7 & \leq 1152 \\
VD: & 0.1x_1 & + & 0.2x_2 & & & + & 0.3x_4 & & & + & 0.6x_6 & & & \leq 768 \\
HD: & 0.2x_1 & & & + & 0.8x_3 & & & & & & & + & 0.6x_7 & \leq 1152 \\
BR: & 0.05x_1 & + & 0.03x_2 & & & + & 0.07x_4 & + & 0.1x_5 & & & + & 0.08x_7 & \leq 384 \\
PL: & & & & & 0.01x_3 & & & + & 0.05x_5 & & & + & 0.05x_7 & \leq 384 \\
\end{array}
$$
$$x_1 \leq 500,\ x_2 \leq 1000,\ x_3 \leq 300,\ x_4 \leq 300,\ x_5 \leq 800,\ x_6 \leq 200,\ x_7 \leq 100$$

The single-period problems for the other months would be similar apart from different market bounds, and different capacity figures for the different types of machine.
The matrix has no special structure, the coefficients are not just $\{-1, 1, 0\}$ as in a TUM matrix and non zeros can appear everywhere. The matrix is not necessarily sparse.

Task 2 Model the multi-period (from January to June) factory planning problem as an LP problem. Use mathematical notation and indicate in general terms how many variables and how many constraints your model has.

Solution:
It is necessary to distinguish for each month the quantities of each product manufactured from the quantities sold and held over in storage. These quantities must be represented by different variables. Let the quantities of product i manufactured, sold, and held over in successive months t be represented by variables x_{it}, s_{it}, h_{it}, $t = 1, \dots, 6$.
A convenient way to represent the link between these variables is shown in Figure 1. Hence, the mass balance constraints to be imposed are:

$$h_{i,t-1} + x_{it} = s_{it} + h_{it}$$

Initially (month 0), there is nothing held in stock but finally (month 6) there are (at least) 50 of each product held. This relation involving product 1 gives rise to the following constraints:

$$
\begin{array}{rrrrl}
 & x_{11} - s_{11} & -h_{11} & = 0 \\
h_{11} & +x_{12} - s_{12} & -h_{12} & = 0 \\
h_{12} & +x_{13} - s_{13} & -h_{13} & = 0 \\
h_{13} & +x_{14} - s_{14} & -h_{14} & = 0 \\
h_{14} & +x_{15} - s_{15} & -h_{15} & = 0 \\
h_{15} & +x_{16} - s_{16} & & = 50
\end{array}
$$

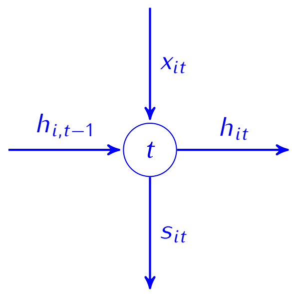


Figure 1: Mass balance constraint at each time period.

Similar constraints must be specified for the other six products. It may be more convenient to define also variables h_{16}, h_{26}, etc, and fix them at the value 50.
The general model is:

$$\max \sum_{i=1}^{7}\sum_{t=1}^{6} p_i s_{it} - \sum_{i=1}^{7}\sum_{t=1}^{6} f h_{it} \quad (1)$$

$$\sum_{i} a_{ij} x_{it} \leq 384(c_j - m_{j,t}) \qquad j \in \{GR, VD, HD, BR, PL\}, t = 1\dots, 6 \quad (2)$$

$$h_{i,t-1} + x_{it} - s_{it} - h_{it} = 0 \qquad i = 1,\dots,7; t = 1,\dots,6 \quad (3)$$

$$s_{it} \leq u_{it} \qquad i = 1,\dots,7; t = 1,\dots,6 \quad (4)$$

$$h_{it} \leq 100 \qquad i = 1,\dots,7; t = 1,\dots,6 \quad (5)$$

$$s_{it}, x_{it}, h_{it} \geq 0 \qquad i = 1,\dots,7; t = 1,\dots,6 \quad (6)$$

$$h_{i0} = 0, h_{i6} = 50 \qquad i = 1,\dots,7 \quad (7)$$

In the objective function (1) the "selling" variables are given the appropriate "unit profit" p_i and the "holding" variables the coefficients of $f = 0.5$. Constraints (2) are the resource constraints where c_m is the capacity for each resource m. Constraints (3) are the mass balance constraints described above and constraints (4) are the marketing limitations where u_{it} are product upper bounds.
The resulting model has the following dimensions:

$6 \times 7 = 42$	manufacturing variables
$6 \times 7 = 42$	selling variables
$6 \times 7 = 42$	holding variables
Total 126	variables
$6 \times 5 = 30$	capacity constraints
$6 \times 7 = 42$	monthly linking constraints
$6 \times 7 = 42$	marketing limitations
$6 \times 7 = 42$	holding quantity constraints
Total 156	constraints

We typically do not count positivity constraints, as those are standard.

If we present the problem in a diagrammatic form we obtain the illustration on the left of Figure 2.
The matrix is not apparently TUM. It has however a *block angular structure*. A *block angular structure* is made by common rows and blocks in diagonal representing submodels. In our case the common rows are the linking equality constraints of mass balance while the submodels are the per period production planning as the one seen in Task 1. Clearly, a matrix with *block angular structure* without common constraints could be decomposed and each submodel solved separately. Nevertheless advanced techniques exist to handle efficiently problems with block angular structure. A typical problem with this structure often used in examples is the multi-commodity flow problem (we will see this in one of the next classes).
Another type of structure which may arise in multi-period models is the *staircase* structure which is illustrated in Figure 2, right. In fact a staircase structure such as this could be converted into a block

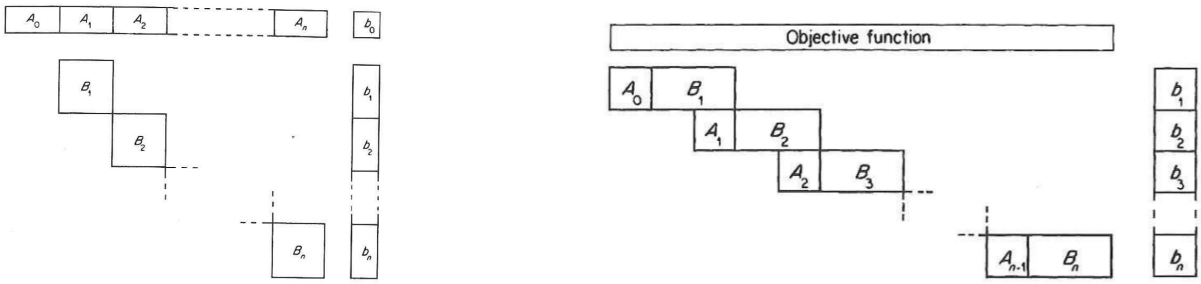


Figure 2: On the left a block angular structure and on the right a staircase structure

angular structure. If alternate "steps" such as (A_0, B_1), (A_2, B_3) were treated as subproblem constraints and the intermidiate "steps" as common rows we would have a block angular structure.