March 22, 2020
Marco Chiarandini

DM545/DM871 – Linear and integer programming

Exercise Sheet, Week 13, Spring 2020 [pdf format]

Solution:
Included.

Exercise 1* MILP Modeling

Manpower Planning. Given a set of workers and the need to cover a set of 15 working hours per day with a, possibly different, number of required persons as staff at each hour, decide the staff at each hour taking into consideration that each person works in shifts that cover 7 hours and hence a person starting in hour i contributes to the workload in hours $i, \ldots, i+6$ (e.g., a person starting in hour 3 contributes to the workload in hours 3,4,5,6,7,8,9).
Formulate the problem to determine the number of people required to cover the workload in mathematical programming terms.

Solution:
Decision Variables:

- $x_i \in \mathbb{Z}_0^+$: number of people starting work in hour $i (i = 1, \ldots, 15)$

Objective Function:

$$\min \sum_{i=1}^{15} x_i$$

Constraints:

- Demand:

$$\sum_{i=t-6}^{i=t} x_i \geq d_t \text{ for } t = 1, \ldots, 15$$

  The number of workers at time t is given by those that started at $t, t-1, \ldots, t-6$

- Bounds:

$$x_{-5}, \ldots, x_0 = 0$$

- Variables:

$$x_i \in \mathbb{Z}_0^+ \qquad i = 1, \ldots, 15$$

Shift scheduling. The administrators of a department of a urban hospital have to organize the working shifts of nurses maintaining sufficient staffing to provide satisfactory levels of health care. Staffing requirements at the hospital during the whole day vary from hour to hour and are reported in Table 1. According to union agreements, nurses can work following one of the seven shift patterns in Table 2 each with its own cost.
The department administrators would like to identify the assignment of nurses to working shifts that meets the staffing requirements and minimizes the total cost.

Solution:

Exercise 2*

Consider the following three matrices:

Hour	Staffing requirement
0 am to 6 am	2
6 am to 8 am	8
8 am to 11 am	5
11 am to 2 pm	7
2 pm to 4 pm	3
4 pm to 6 pm	4
6 pm to 8 pm	6
8 pm to 10 pm	3
10 pm to 12 pm	1

Table 1:

pattern	Hours of work	total hours	cost
1	0 am to 6 am	6	720 Dkk
2	0 am to 8 am	6	800 Dkk
3	6 am to 2 pm	8	740 Dkk
4	8 am to 4 pm	8	680 Dkk
5	2 pm to 10 pm	8	720 Dkk
6	4 pm to 12 pm	6	780 Dkk
7	6 pm to 12 pm	6	640 Dkk

Table 2:

$$
\begin{array}{lll}
\min & 720x_1 + 800x_2 + 740x_3 + 680x_4 + 720x_5 + 780x_6 + 640x_7 & \\
0-6: & x_1 + x_2 & \geq 2 \\
6-8: & x_2 + x_3 & \geq 8 \\
8-11: & x_3 + x_4 & \geq 5 \\
11-14: & x_3 + x_4 & \geq 7 \\
14-16: & x_4 + x_5 & \geq 3 \\
16-18: & x_5 + x_6 & \geq 4 \\
18-20: & x_5 + x_6 + x_7 & \geq 6 \\
20-22: & x_5 + x_6 + x_7 & \geq 3 \\
22-24: & x_6 + x_7 & \geq 1 \\
 & x_1, x_2 \dots x_7 \geq 0 \text{ and integer.} &
\end{array}
$$

$$
\begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}
\qquad
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}
\qquad
\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}
$$

For each of them say if it is totally unimodular and justify your answer.

Solution:

We look for the satisfaction of the conditions of the theorem saw in class. Accordingly, it is *sufficient* for a matrix to be TUM to find a partition of the rows such that the ones with same sign are in different partitions and those with different sign in the same partition.

The first matrix is TUM. The partition is $I_1 = \{1, 4\}$ and $I_2 = \{2, 3\}$.
The second matrix is TUM. The partition is $I_1 = \{1, 2, 3\}$ and $I_2 = \{4\}$.
The third matrix is not TUM. Here the theorem does not apply and there is a submatrix with deteminant -2, hence we cannot be sure that the solutions associated with the matrix will be integer.

Exercise 3*

In class, we proved that the (mininum) vertex covering problem and the (maximum) matching problem are a weak dual pair. Prove that for bipartite graphs they, actually, are a strong dual pair.

Solution:

The formulation of the matching problem is:

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ & \sum_{e \in E: v \in e} x_e \leq 1 \qquad \forall v \in V \\ & x_e \in \{0, 1\} \quad \forall e \in E \end{aligned}$$

If we take the linear relaxation and make the dual of it we obtain:

$$\begin{aligned} \min \quad & \sum_{v \in V} y_v \\ & y_v + y_u \geq w_{uv} \quad \forall u, v \in V, uv \in E \\ & y_v \geq 0 \quad \forall v \in V \end{aligned}$$

This latter is the linear relaxation of the vertex cover problem.
Hence the two problems make a weak dual pair. It is not strong, indeed a triangle has optimal vertex cover 2 and optimal matching 1.
In bipartite graphs instead the pair is strong dual. Indeed, the solution of the linear relaxation of the matching problem on bipartite graphs is always integer. The same must old for the dual and hence the gap is closed.
A more formal proof is on page 24-25 of [Wo].

Exercise 4*

Generalized Assignment Problem. Suppose there are n types of tracks available to deliver products to m clients. The cost of track of type i serving client j is c_{ij}. The capacity of track type i is Q_i and the demand of each client is d_j. There are a_i tracks for each type. Formulate an IP model to decide how many tracks of each type are needed to satisfy all clients so that the total cost of doing the deliveries is minimized. If all the input data will be integer, will the solution to the linear programming relaxation be integer?

Solution:

It is good trying to model the problem as a min cost flow problem. However, one can soon realize that this is not possible since we are asked for the number of tracks but we need to take into account a demand and a capacity of products.
We can however write an ILP model:

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{1}$$

$$\sum_{j \in J} x_{ij} \leq a_i \qquad \forall i \in I \tag{2}$$

$$\sum_{i \in I} Q_i x_{ij} \geq d_j \qquad \forall j \in J \tag{3}$$

$$x_{ij} \geq 0 \text{ and integer} \qquad \forall (i, j) \in A \tag{4}$$

The solutions of the linear relaxation are not necessarily integer, becasue this is not a min cost flow model and the matrix is not trivially TUM.

Exercise 5

1. In class we stated that for the uncapaciteted facility location problem there are two formulations:

$$X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^{m} x_i \leq my, x_i \leq 1 \text{ for } i = 1, \dots, m\}$$

$$P = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \dots, m, y \leq 1\}$$

Prove that the polyhedron $P = \{(x_1, \dots, x_m, y) \in \mathbb{R}^{m+1} : y \leq 1, x_i \leq y \text{ for } i = 1, \dots, m\}$ has integer vertices. [Hint: start by writing the constraint matrix and show that it is TUM.]

Solution:

(a)

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & y \\ & & & & & 1 \\ & & 1 & & & -1 \\ & 1 & & & & -1 \\ 1 & & & & & -1 \\ & & & 1 & & -1 \\ & & & & 1 & -1 \end{bmatrix} \begin{array}{c} \\ \leq 1 \\ \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \end{array}$$

$$A^T \text{ is } \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & -1 & -1 & -1 & -1 \end{bmatrix} \Rightarrow \text{TUM for } I_1 = \text{all...}, \; I_2 = \emptyset$$

$$\sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij} = 0 \quad \forall j$$

2. Consider the following (integer) linear programming problem:

$$\begin{aligned} \min \; & c_1x_1 + c_2x_2 + c_3x_3 + c_3x_4 \\ & x_3 + x_4 \geq 10 \\ & x_2 + x_3 + x_4 \geq 20 \\ & x_1 + x_2 + x_3 + x_4 \geq 30 \\ & x_2 + x_3 \geq 15 \\ & x_1, x_2, x_3, x_4 \in \mathbb{Z}^+_0 \end{aligned} \tag{5}$$

The constraint matrix has consecutive 1's in each column. Matrices with consecutive 1's property for each column are totally unimodular. Show that this fact holds for the specific numerical example (5). That is, show first that the constraint matrix of the problem has consecutive 1s in the columns and then that you can transform this matrix into one that you should recognize to be a TUM matrix. [Hint: rewrite the problem in standard form (that is, in equation form) and add a redundant row $\mathbf{0} \cdot \mathbf{x} = 0$ to the set of constraints. Then perform elementary row operations to bring the matrix to a TUM form.]

Solution:

b)

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 20 \\ 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 30 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for each $i = 4, 3 .. 1$ subtract ith row with $(i+1)$th row

$$\begin{array}{ccccccccc} 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 10 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 10 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -15 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -15 \end{array}$$

$\Rightarrow$ TUM and min cost flow

3. Use one of the two previous results to show that the *shift scheduling problem* in Exercise 1 of this Sheet can be solved efficiently when formulated as a mathematical programming problem. (You do not need to find numerical results.)

Solution:

c) min $720x_1 + 800x_2 + 740x_3 + 680x_4 + 720x_5 + 780x_6 + 640x_7$

0 - 6: $x_1 + x_2 \geq 2$

6 - 8: $x_2 + x_3 \geq 8$

8 - 11: $x_3 + x_4 \geq 5$

11 - 14: $x_3 + x_4 \geq 7$

14 - 16: $x_4 + x_5 \geq 3$

16 - 18: $x_5 + x_6 \geq 4$

18 - 20: $x_5 + x_6 + x_7 \geq 6$

20 - 22: $x_5 + x_6 + x_7 \geq 3$

22 - 24: $x_6 + x_7 \geq 1$

$x_1, x_2 \dots x_7 \geq 0$ and integer.

The matrix has consecutive 1's property on cols hence LP relax gives integer results.

Exercise 6* Network Flows: Problem of Representatives

A town has r residents $R_1, R_2, \ldots, R_r$; q clubs $C_1, C_2, \ldots, C_q$; and p political parties $P_1, P_2, \ldots, P_p$. Each resident is a member of at least one club and can belong to exactly one political party. Each club must nominate one of its members to represent it on the town's governing council so that the number of council members belonging to the political party P_k is at most u_k. Is it possible to find a council that satisfies this "balancing" property?
Show how to formulate this problem as a maximum flow problem.

Solution:

[AMO] at library bookshelf pages 170-176.

Exercise 7* Scheduling on Uniform Parallel Machines

We consider scheduling a set J of jobs on M uniform parallel machines. Each job $j \in J$ has a processing requirement p_j (denoting the number of machine days required to complete the job), a release data r_j (representing the beginning of the day when job j become available for processing), and a due date $d_j \geq r_j + p_j$ (representing the beginning of the day by which the job must be completed). We assume that a machine can work on only one job at a time and that each job can be processed by at most one machine at a time. However we allow preemptions (ie, we can interrupt a job and process it on different machines on different days). The scheduling problem is to determine a feasible schedule that completes all jobs before their due dates or to show that no such schedule exists.
Formulate the feasible scheduling problem as a maximum flow problem.

Solution:

[AMO] pages 170-176.

Exercise 8* Tanker Scheduling Problem

A steamship company has contracted to deliver perishable goods between several different origin-destination pairs. Since the cargo is perishable the customers have specified precise dates (ie, delivery dates) when the shipments must reach their destinations. (The cargoes may not arrive early or late). The steamship company wants to determine the minimum number of ships needed to meet the delivery dates of the shiploads.
Formulate this problem as a maximum flow problem modeling the example in Table 3 with four shipments. Each shipment is a full shipload with the characteristics shown in Table 3. For example, as specified by the first row in this figure, the company must deliver one shipload available at port A and destined for port C on day 3.

shipment	origin	destination	delivery date
1	Port A	Port C	3
2	Port A	Port C	8
3	Port B	Port D	3
4	Port B	Port C	6

	C	D
A	3	2
B	2	3

	A	B
C	2	1
D	1	2

Table 3: Data for the tanker scheduling problem: Left shipment characteristics; Center, shipment transit times; Right return times.

Solution:

[AMO] pages 170-176.

Exercise 9* Directed Chinese Postman Problem

Suppose a postman has to deliver mail along all the streets in a small town. Assume furthermore that on one-way streets the mail boxes are all on one side of the street, whereas for two-way streets, there are mail boxes on both sides of the street. For obvious reasons the postman wishes to minimize the distance he has to travel in order to deliver all the mail and return home to his starting point. Show how you can solve this problem using minimum cost flows. A similar model can be formulated for the Snow Plow problem (`http://city.temeda.com/`) or the Salt Spreding problem.

Solution:

The solution to this problem can be found on page 174 of J. Bang-Jensen and G. Gutin. Digraphs: Theory, Algorithms and Applications, Springer London, 2009 http://dx.doi.org/10.1007/978-1-84800-998-1_4.

In short, the solution to the Chinese postman problem is a min cost flow that traverses each arc at least once in a network where arcs are streets and nodes are street instersections.

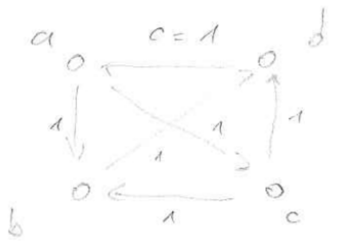
We must assume that the graph is strongly connected otherwise there is no solution (any closed walk is strongly connected.)

A digraph where there exists a walk that visits arcs exactly once is called Eulerian.

$$N = (V, A, \mathbf{l}, \mathbf{u} = \infty, \mathbf{c})$$

$$x_{ij} = \#\text{ of times arc is used}$$

The cost os a min cost circulation in N equals the cost of a Chinese postman walk in D.

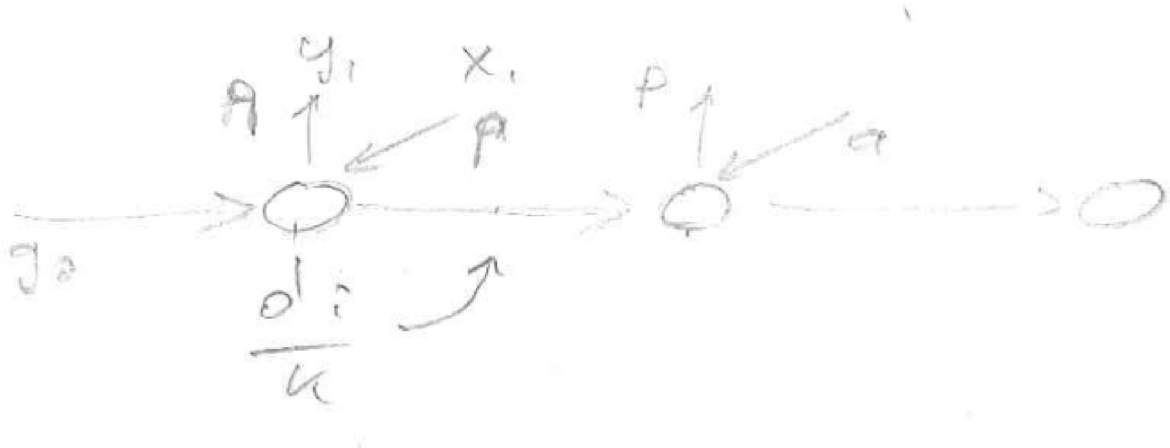


Exercise 10

The production plan of a factory for the next year is to produce d_t units of product per month t, $t = 1, \dots, 12$. Each worker can produce k units of product in a month. The monthly salary is equal to s. Employing and firing personnel has costs: precisely, employing one person costs p while firing one costs q. Assuming that initially there are g_0 workers, determine the number of workers that must be present during every month such that the demand is always satisfied and the overall costs of salary, employment, and firing are minimized.

Solution:

It is possible to model the employment and firing of workers as a flow in a network.

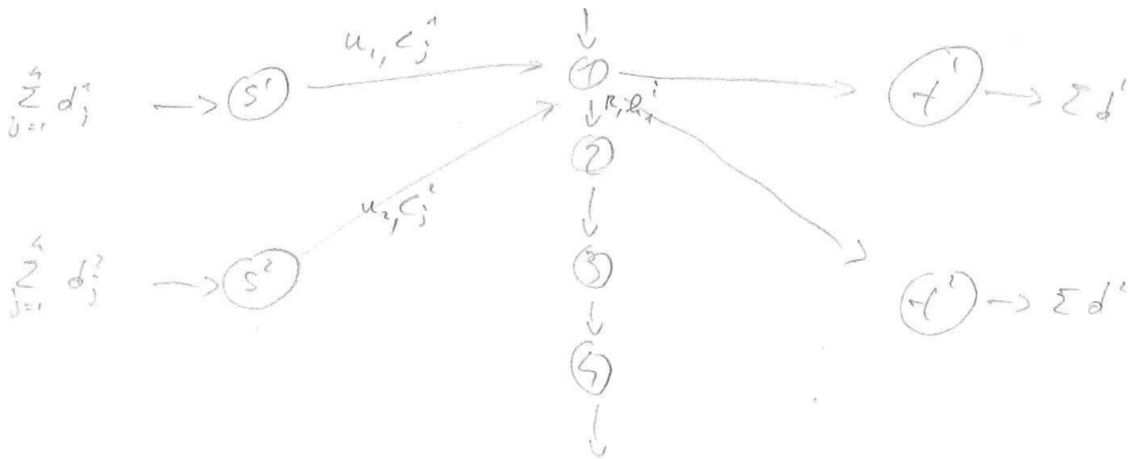


Exercise 11 Warehousing of Seasonal Products

A company manufactures multiple products. The products are seasonal with demand varying weekly, monthly, or quarterly. To use its work-force and capital equipment efficiently, the company wishes to "smooth" production, storing pre-season production to supplement peak-season production. The company has a warehouse with fixed capacity R that it uses to store all the products it produces. Its decision problem is to identify the production levels of all the products for every week, month, or quarter of the year that will permit it to satisfy the demands incurring the minimum possible production and storage costs.
We can represent this warehousing problem as a relevant generalization of the min cost network flow problem encountered in the course.
For simplicity, consider a situation in which the company makes two products and then it needs to schedule its production for each of the next four quarters of the year. Let d_j^1 and d_j^2 denote the demand for products 1 and 2 in quarter j. Suppose that the production capacity for the jth quarter is u_j^1 and u_j^2, and that the per unit cost of production for this quarter is c_j^1 and c_j^2. Let h_j^1 and h_j^2 denote the storage (holding) costs per unit of the two products from quarter j to quarter $j+1$.
Represent graphically the network in the two products four periods case and write the Linear Programming formulation of the problem. Which network flows problem models this application? If all input data are integer, will the solution be integer?

Solution:

We can model this problem as a multicommodity flow in a network. The network has 4 nodes, one for each quarter, two target nodes for each quarter (one per product), two nodes for each plant and for each quarter (one per product) and two global sources for the two products.



See page 655 of [AMO].

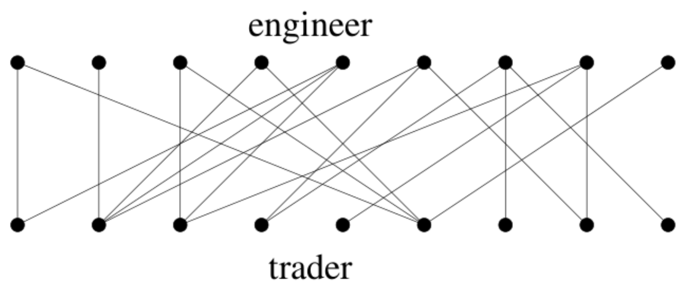


Figure 1:

Exercise 12

A managing director has to launch the marketing of a new product. Several candidate products are at his disposal and he has to choose the best one. Hence, he let each of these products be analysed by a team made of an engineer and a trader who write a review together. The teams are made along the graph in Figure 1; each edge corresponds to a product and its endvertices to the engineer and trader examining it.

a) How many people at least does the managing director gather in order to have the report on all the products? (The report can be given by either the engineer or the trader.)

b) Assuming now that the report must be done jointly by an engineering and a trader, and that each engineer and trader can be occupied with only one candidate product, give a polynomial time algorithm to identify which products will for sure not have the possibility to obtain a report.

Solution:

a) This is an application of the vertex cover problem and its strong duality with maximum matching.
b) This can be done by finding all maximum matching of the graph. The edges that are never in a matching are those that will be never reviewed. We could solve $|E|$ linear programs formulations of the max matching problem for bipartite graphs in each of which a different edge is enforced to be in the solution. Other, more efficient methods based on direct algorithms exist.

References