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DM545/DM871 – Linear and integer programming

Sheet 4, Spring 2020 [pdf format]

Starred exercises are relevant for the tests.

Exercise 1*

Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & z = x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- Solving an an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. The solution of the auxiliary LP problem gives an initial feasible basis or a proof of infeasibility.
- Applying the dual simplex on a possibly modified problem to find a feasible solution. If the initial infeasible tableau of the original problem is not optimal then the objective function can be temporarily modified for this phase in order to make the initial tableau optimal although not feasible. Opposite to the primal simplex method, the dual simplex method iterates through infeasible basis solutions, while maintaining them optimal, and stops when a feasible solution is reached.

Dual Simplex: The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:

1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| \ : \ a_{i^*j} < 0\}$
3. Update the tableau around the pivot in the same way as with the primal simplex.
4. Stop if no right-hand side term is negative.

Duality can help us with the issue of initial feasible basis solutions. In the problem above, if the objective function was $w = -x_1 - x_2$, then the initial basis solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. But with objective function z the simplex has infeasible initial basis in both problems. However we can change temporarily the objective function z with w and apply the dual simplex method. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and continue iterating with the primal simplex. The phase I–phase II simplex method that uses the dual simplex is also called the *dual-primal simplex method*.
Apply this method to the problem above and verify that it leads to the same solution as in point 1.

Exercise 2* Sensitivity Analysis and Revised Simplex

A furniture-manufacturing company can produce four types of product using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.
- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.
- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve the following LP problem

$$\begin{aligned} \max \quad & 19x_1 + 13x_2 + 12x_3 + 17x_4 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 \\ & x_1 + x_2 + x_3 + x_4 \leq 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The final tableau has x_1, x_3 and x_4 in basis. With the help of a computational environment such as Python for carrying out linear algebra operations, address the following points:

a) Write A_B, A_N, $A_B^{-1}A_N$, the final simplex tableau and verify that the solution is indeed optimal.

b) What is the increase in price (reduced cost) that would make product x_2 worth to be produced?

c) What is the marginal value (shadow price) of an extra hour of work or amount of metal and wood?

d) Are all resources totally utilized, i.e. are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

e) From the economical interpretation of the dual why product x_2 is not worth producing? What is its imputed cost?

Solve the following variations:

1. The net profit brought in by each desk increases from 13 Euro to 15 Euro.
2. The availability of metal increases from 117 to 125 units per day
3. The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood and bring in a net profit of 14 Euro.
4. The number of chairs produced must be at most five times the numbers of desks

Exercise 3

Solve the systems $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1\ 2\ 3]$ and $E_1 E_2 E_3 E_4 \mathbf{d} = [1\ 2\ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Exercise 4*

Write the dual of the following problem

$$
\begin{aligned}
(P) \qquad & \max \sum_{j\in J}\sum_{i\in I} r_j x_{ij} \\
& \sum_{j\in J} x_{ij} \leq b_i && \forall i \in I \\
& \sum_{i\in I} x_{ij} \leq d_j && \forall j \in J \\
& \sum_{i\in I} p_i x_{ij} = p_j \sum_{i\in I} x_{ij} && \forall j \in J \\
& x_{ij} \geq 0 && \forall i \in I, j \in J
\end{aligned}
$$

Exercise 5 Factory Planning and Machine Maintenance

A firm makes seven products $1, \dots, 7$ on the following machines: 4 grinders, 2 vertical drills, 3 horizontal drills, 1 borer, and 1 planer.
Each product yields a certain contribution to the profit (defined as selling price minus cost of raw materials expressed in Euro/unit). These quantities (in Euro/unit) together with the production times (hours/unit) required on each process are given below.

product	1	2	3	4	5	6	7
profit	10	6	8	4	11	9	3
grinding	0.5	0.7	0	0	0.3	0.2	0.5
vdrill	0.1	0.2	0	0.3	0	0.6	0
hdrill	0.2	0	0.8	0	0	0	0.6
boring	0.05	0.03	0	0.07	0.1	0	0.08
planning	0	0	0.01	0	0.05	0	0.05

In the first month (January) and the five subsequent months certain machines will be down for maintenance. These machines will be:

January	1 grinder
February	2 hdrill
March	1 borer
April	1 vdrill
May	1 grinder
May	1 vdrill
June	1 planer
June	1 hdrill

There are marketing limitations on each product in each month. That is, in each month the amount sold for each product cannot exceed these values:

product	1	2	3	4	5	6	7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

It is possible to store products in a warehouse. The capacity of the storage is 100 units per product type per month. The cost is 0.5 Euro per unit of product per months. There are no stocks in the first month but it is desired to have a stock of 50 of each product type at the end of June.
The factory works 6 days a week with two shifts of 8 hours each day. (It can be assumed that each month consists of 24 working days.)
The factory wants to determine a production plan, that is, the quantity to produce, sell and store in each month for each product, that maximizes the total profit.

Task 1 Model the factory planning problem for the month of January as an LP problem.

Task 2 Model the multi-period (from January to June) factory planning problem as an LP problem. Use mathematical notation and indicate in general terms how many variables and how many constraints your model has.