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DM545/DM871 – Linear and integer programming

Sheet 5, Spring 2020 [pdf format]

Exercise 1*

(The following is exercise 6 from exam 2010) A car rental company at the beginning of each month wants to have a certain number of cars in each of the towns in which it operates. For the towns $A, B, C, \ldots, G$ the number of cars desired is $30, 40, 55, 60, 80, 40, 55$, respectively. At the end of the current month there are instead in the stations in these towns $65, 90, 95, 15, 60, 10, 25$ cars, respectively. To move one car from one station to the other causes a cost that we may assume proportional to the distance between the two stations. The table indicates the distances (in hundreds of kilometers) between every city pair of stations.

from .. to..	D	E	F	G
A	5	6	10	9
B	9	11	9	15
C	12	10	14	15

Formulate the problem of deciding the cars to move while minimizing the costs in mathematical programming terms. Which algorithm could you use to solve the problem beside the simplex?

Exercise 2*

Formulate the following manpower planning problem in mathematical programming terms.
Given a set of workers and a set of 15 working hours per day with a required staffing per hour. Determine the minimum number of people required to cover the workload requirements, knowing that a person works in shifts that cover 7 hours and a person starting in hour i contributes to the workload in hours $i, \ldots, i+6$ (eg: a person starting in hour 3 contributes to the workload in hours 3,4,5,6,7,8,9).

Exercise 3* Cutting-Stock Problem

Materials such as paper, textiles, cellophane, and metallic foil are manufactured in rolls of large widths. These rolls, referred to as *raws*, are later cut into rolls of small width, called *finals*. Each manufacturer produces raws of a few standard widths; the widths of the finals are specified by different customers and may vary widely. The cutting is done on machines by knives that slice through the rolls in much the same way as a knife slices bread. For example, a raw that is 100 cm wide may be cut into two finals with 31 cm width and one final with 36 cm width, with the 2 cm left over going to waste. When a complicated summary of orders has to be filled, the most economical way of cutting the existing raws into the desired finals is rarely obvious. The problem of finding such a way is known as the *cutting-stock problem*.
Model the problem in mathematical programming terms.

Exercise 4

A company producing and selling electricity has to plan the production for the next n days. It can choose among m types of fuels, e.g., cool, biomass, wind, etc.
A production unit can be on or off and this status can change on hourly basis. If a production unit uses fuel i, it generates a_i kilowatt of electricity at the cost of c_i. Due to limits on CO_2 emissions, the maximum number of production units per day burning the same fuel i equals to r_i.
The decision maker has to decide the type of fuel to use for each production unit on each hour for the next n days.
The further constraints apply: each fuel i can be used in more than one production unit per day but it can be used in at most p_i days; the costs of production per day must be non increasing during the time horizon, while the daily usage time must be non-decreasing during the production horizon. Let A be a

set of pair of fuels. In each day and for each pair of fuels (h, k), if fuel h is used in at least one unit, then fuel k must be planned in at least a unit in the same day as well.
The objective is to maximize the electricity produced over the time horizon.
Model the problem as a MIP problem.

Exercise 5*

Write the following logical conditions as linear conditions on 0-1 variables:

$$\begin{aligned} X_1 \vee X_2 \\ X_1 \rightarrow X_2 \\ X_1 \wedge X_2 \\ X_1 \leftrightarrow X_2 \\ \neg X_1 \\ X_1 \oplus X_2 \text{ xor} \end{aligned}$$

Exercise 6*

Formulate the satisfiability problem as an ILP. Give the precise formulation for the following instance:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_2 \vee \neg x_3) \vee (x_3 \vee \neg x_1) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

Exercise 7*

Model by means of MIP the following constraints:

- at least $h < k$ of the linear constraints $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, \ldots, k$ are satisfied.
- $z = xy$, where $x, y \in \mathbb{B}$.

Exercise 8

Consider the following mathematical programming problem:

$$\max\{\sum_{i,j:i<j} c_{ij} z_i z_j : z \in B^n\}$$

What type of programming problem is it? Put it in a form that we can solve with the methods seen so far in the course.

Exercise 9*

Reformulate the following problem into a 0-1 ILP:

$$\begin{aligned} \max x_1^3 + x_2^5 + x_3^2 + 5x_1x_2x_3^4 - 2x_1x_2 \\ 7x_1x_2^2 + 3x_1^2x_2 - 5x_1x_2 \geq 0 \\ -2x_1 + 3x_2 + x_3 \leq 3 \\ x_i \in \{0, 1\}, i = 1, 2, 3 \end{aligned}$$

Exercise 10*

Show that the maximum matching in arbitrary graphs is a special sort of set packing problem.