DM545/DM871

Linear and Integer Programming

# More on Polyhedra and Farkas Lemma

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# Outline

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We now look at Farkas Lemma with two objectives:

- (giving another proof of strong duality)
- understanding a certificate of infeasibility

# Farkas Lemma

**Theorem (Farkas' Lemma)**

*Let* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *Then,*

$$\begin{array}{rll} \textit{either} & I. & \exists x \in \mathbb{R}^n : Ax = b \textit{ and } x \geq 0 \\ \textit{or} & II. & \exists y \in \mathbb{R}^m : y^T A \geq 0^T \textit{ and } y^T b < 0 \end{array}$$

Easy to see that both I and II cannot occur together:

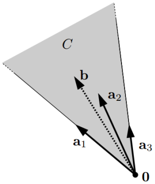
$$(0 \leq) \qquad y^T A x = y^T b \qquad (< 0)$$

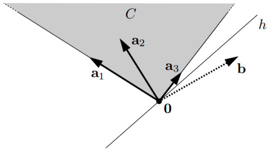
# Geometric interpretation of Farkas Lemma

Linear combination of $a_i$ with nonnegative terms generates a convex cone:

$$\{\lambda_1 a_1 + \ldots + \lambda_n a_n, | \lambda_1, \ldots, \lambda_n \geq 0\}$$

Polyhedral cone: $C = \{x \mid Ax \leq 0\}$, intersection of many $ax \leq 0$
Conic hull of rays $p_i = \{\lambda_i a_i, \lambda_i \geq 0\}$




Either point $b$ lies in convex cone $C$

or $\exists$ hyperplane $h$ passing through point 0 $h = \{x \in \mathbb{R}^m : y^T x = 0\}$ for $y \in \mathbb{R}^m$ such that all vectors $a_1, \ldots, a_n$ (and thus $C$) lie on one side and $b$ lies (strictly) on the other side (ie, $y^T a_i \geq 0, \forall i = 1 \ldots n$ and $y^T b < 0$).

# Alternative Formulation

**Theorem (Farkas' Lemma)**

*The inequality* $c^T x \geq c_0$ *is valid for the non-empty polyhedron* $P := \{x \geq 0 \mid Ax = b\}$
***if and only if*** $y \in \mathbb{R}^m$ *exists such that:*

$$\begin{aligned} c^T &\geq y^T A \\ c_0 &\leq y^T b \end{aligned}$$

$\Longleftarrow$ (sufficiency) (used in Gomory cuts)

$$c^T x \geq \quad y^T A x = y^T b \quad \geq c_0$$

$\Longrightarrow$ (necessity)
by simplex algorithm similar to our proof of the strong duality theorem

# Other Variants of Farkas Lemma

**Corollary**

(i) $Ax = b$ *has sol* $x \geq 0 \iff \forall y \in \mathbb{R}^m$ *with* $y^T A \geq 0^T, y^T b \geq 0$

(ii) $Ax \leq b$ *has sol* $x \geq 0 \iff \forall y \geq 0$ *with* $y^T A \geq 0^T, y^T b \geq 0$

(iii) $Ax \leq 0$ *has sol* $x \in \mathbb{R}^n \iff \forall y \geq 0$ *with* $y^T A = 0^T, y^T b \geq 0$

# Certificate of Infeasibility

Farkas Lemma provides a way to certificate infeasibility.

**Theorem**

*Let* $Ax = b$, $x \geq 0$.
*Given a certificate* $y^*$ *it is easy to check the conditions (by linear algebra):*

$$\begin{aligned} A^T y^* &\geq 0 \\ by^* &< 0 \end{aligned}$$

Why would $y^*$ be a certificate of infeasibility?
Proof (by contradiction)
Assume, $A^T y^* \geq 0$ and $by^* < 0$.
Moreover assume $\exists x^*$: $Ax^* = b$, $x^* \geq 0$,then:

$$(\geq 0) \qquad (y^*)^T A x^* = (y^*)^T b \qquad (< 0)$$

Contradiction

General form:

$$\begin{aligned} \max\ & c^T x \\ A_1 x &= b_1 \\ A_2 x &\leq b_2 \\ A_3 x &\geq b_3 \\ x &\geq 0 \end{aligned}$$

infeasible $\Leftrightarrow \exists y^*$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 + b_3^T y_3 &> 0 \\ A_1^T y_1 + A_2^T y_2 + A_3^T y_3 &\leq 0 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \end{aligned}$$

**Example**

$$\begin{aligned} \max\ & c^T x \\ x_1 &\leq 1 \\ x_1 &\geq 2 \end{aligned}$$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 &> 0 \\ A_1^T y_1 + A_2^T y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 &> 0 \\ y_1 + y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$y_1 = -1, y_2 = 1$ is a valid certificate.

- Observe that it is not unique!
- It can be reported in place of the dual solution because same dimension.
- To repair infeasibility we should change the primal at least so much as that the certificate of infeasibility is no longer valid.
- Only constraints with $y_i \neq 0$ in the certificate of infeasibility cause infeasibility

# Duality: Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
  - Farkas Lemma:
    Strong duality + Infeasibility certificate
- Dual Simplex
- Economic interpretation
- Geometric Interpretation
- Sensitivity analysis

# Resume

Advantages of considering the dual formulation:

- proving optimality (although the simplex tableau can already do that)
- gives a way to check the correctness of results easily
- alternative solution method (ie, primal simplex on dual)
- sensitivity analysis
- solving P or D we solve the other for free
- certificate of infeasibility

# Outline

# Interior Point Algorithms

- Ellipsoid method: cannot compete in practice but weakly polynomial time (Khachyian, 1979)
- Interior point algorithm(s) (Karmarkar, 1984) competitive with simplex and polynomial in some versions
  - affine scaling algorithm (Dikin)
  - logarithmic barrier algorithm (Fiacco and McCormick) $\equiv$ Karmakar's projective method

1. Start at an interior point of the feasible region
2. Move in a direction that improves the objective function value at the fastest possible rate while ensuring that the boundary is not reached
3. Transform the feasible region to place the current point at the center of it

- because of patents reasons, now mostly known as barrier algorithms
- one single iteration is computationally more intensive than the simplex (matrix calculations, sizes depend on number of variables)
- particularly competitive in presence of many constraints (eg, for $m = 10,000$ may need less than $100$ iterations)
- bad for post-optimality analysis $\rightsquigarrow$ crossover algorithm to convert a solution of barrier method into a basic feasible solution for the simplex

# Outline

1. Farkas Lemma
2. Beyond the Simplex
3. Perspectivization

# Event Announcement


3 Computer Scientists, 3 Math.-Econ., 3 Data Scientist and 3 Math./App.Math.

# MatØk - Operationsanalyse

60 ECTS konstituerende fag:

- DM817 Netværksprogrammering: Teori og anvendelser (10 ECTS, efterår)
- DM841 Heuristikker og constraint programmering for diskret optimering (10 ECTS, efterår)
- Microeconometrics (10 ECTS, efterår)

- DM867 Kombinatorisk optimering (10 ECTS, forår)
- DM879 Kunstig intelligens (10 ECTS, forår)
- ST816 Beregningsmæssig statistik, del 1 (5 ECTS, forår)
- DM872 Matematisk optimering i praksis (5 ECTS, forår)

- 30 ECTS valgfag
- 30 ECTS kandidatspeciale