DM545/DM871

Linear and Integer Programming

Lecture

Cutting Planes

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Outline

Valid Inequalities

- IP: $z = \max\{c^T x : x \in X\}$, $X = \{x : Ax \leq b, x \in \mathbb{Z}^n_+\}$
- Proposition: $\mathrm{conv}(X) = \{x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$ is a polyhedron
- LP: $z = \max\{c^T x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$ would be the best formulation
- $\tilde{a}x \leq \tilde{b}$ facet defining inequalities
- Key idea: try to approximate the best formulation.

Definition (Valid inequalities)

$ax \leq b$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $ax \leq b \; \forall x \in X$

Which are useful inequalities? and how can we find them? How can we use them?

Example: Pre-processing

- $X = \{(x, y) : x \leq 999y; \quad 0 \leq x \leq 5, \quad y \in \mathbb{B}^1\}$

$$x \leq 5y$$

- $X = \{x \in \mathbb{Z}^4_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6 + \frac{6}{11}$$
$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

- Capacitated facility location:

$$\sum_{i \in M} x_{ij} \leq b_j y_j \quad \forall j \in N \qquad\qquad x_{ij} \leq b_j y_j$$
$$\sum_{j \in N} x_{ij} = a_i \quad \forall i \in M \qquad\qquad x_{ij} \leq a_i$$
$$x_{ij} \geq 0, \quad y_j \in B^n \qquad\qquad x_{ij} \leq \min\{a_i, b_j\} y_j$$

Converting Weak to Strong MIP Formulations

Strong formulations $\equiv$ better, tighter formulations

Detection possible from the log output of a solver.

Possible actions:

1. Add cuts to existing models
   - Combining constraints
   - Using a graph representation (clique cuts)
   - Using a disjunctive approach

$\leadsto$ Many found automatically by the solver in pre-solver phase

2. (Change the model)

3. (Change the algorithm, eg, column generation)

(Lazy) constraints $\neq$ cuts

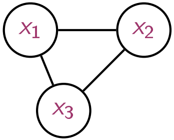
Add cuts to the existing model

$$\begin{aligned} \text{maximize } & x_1 + x_2 + x_3 \\ \text{subject to } & x_1 + x_2 \leq 1 \\ & x_2 + x_3 \leq 1 \\ & x_1 + x_3 \leq 1 \\ & x_i \in \{0, 1\} \qquad i = 1, 2, 3 \end{aligned}$$

Combine and round constraints:

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &\leq 3 \\ x_1 + x_2 + x_3 &\leq \frac{3}{2} \\ x_1 + x_2 + x_3 &\leq 1 \end{aligned}$$

Create a conflict graph; at most one binary in a clique can be 1



$$x_1 + x_2 + x_3 \leq 1$$

Chvátal-Gomory cuts

- $X \in P \cap \mathbb{Z}^n_+, \quad P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}, \quad A \in \mathbb{R}^{m \times n}$
- $u \in \mathbb{R}^m_+$, $\{a_1, a_2, \ldots a_n\}$ columns of A

CG procedure to construct valid inequalities

1) $\sum_{j=1}^{n} u^T a_j x_j \leq u^T b$ valid: $u \geq 0$

2) $\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \leq u^T b$ valid: $x \geq 0$ and $\sum \lfloor u^T a_j \rfloor x_j \leq \sum u^T a_j x_j$

3) $\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \leq \lfloor u^T b \rfloor$ valid for X since $x \in \mathbb{Z}^n$

Theorem

by applying this CG procedure a finite number of times every valid inequality for X can be obtained

However not all the constraints generated by $u \in \mathbb{R}^m_+$ are tightenings.

Cutting Plane Algorithms

- $X \in P \cap \mathbb{Z}^n_+$
- a family of valid inequalities $\mathcal{F} : \mathrm{a}^T\mathrm{x} \leq b, (\mathrm{a}, b) \in \mathcal{F}$ for X
- we do not find them all a priori, only interested in those close to optimum

Cutting Plane Algorithm

Init.: $t = 0, P^0 = P$

Iter. t: Solve $\bar{z}^t = \max\{\mathrm{c}^T\mathrm{x} : \mathrm{x} \in P^t\}$
let x^t be an optimal solution
if $\mathrm{x}^t \in \mathbb{Z}^n$ stop, x^t is opt to the IP
if $\mathrm{x}^t \notin \mathbb{Z}^n$ solve separation problem for x^t and $\mathcal{F}$
if (a^t, b^t) is found with $\mathrm{a}^t\mathrm{x}^t > b^t$ that cuts off x^t

$$P^{t+1} = P \cap \{\mathrm{x} : \mathrm{a}^i\mathrm{x} \leq b^i, i = 1, \ldots, t\}$$

else stop (P^t is in any case an improved formulation)

Gomory's fractional cutting plane algorithm

Cutting plane algorithm + Chvátal-Gomory cuts

- $\max\{c^T x : Ax = b, x \geq 0, x \in \mathbb{Z}^n\}$
- Solve LPR to optimality

$$
\left[\begin{array}{c:c:c:c}
I & \bar{A}_N = A_B^{-1} A_N & 0 & \bar{b} \\
\hdashline
\bar{c}_B & \bar{c}_N (\leq 0) & 1 & -\bar{d}
\end{array}\right]
$$

$$
\begin{aligned}
x_{B_u} &= \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j, \quad u = 1..m \\
z &= \bar{d} + \sum_{j \in N} \bar{c}_j x_j
\end{aligned}
$$

- If basic optimal solution to LPR is not integer then $\exists$ some row u: $\bar{b}_u \notin \mathbb{Z}^1$.
  The Chvatál-Gomory cut applied to this row is:

$$
x_{B_u} + \sum_{j \in N} \lfloor \bar{a}_{uj} \rfloor x_j \leq \lfloor \bar{b}_u \rfloor
$$

(B_u is the index in the basis B corresponding to the row u) (cntd)

- Eliminating $x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j$ in the CG cut we obtain:

$$\sum_{j \in N} (\underbrace{\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor}_{0 \le f_{uj} < 1}) x_j \ge \underbrace{\bar{b}_u - \lfloor \bar{b}_u \rfloor}_{0 < f_u < 1}$$

$$\sum_{j \in N} f_{uj} x_j \ge f_u$$

$f_u > 0$ or else u would not be row of fractional solution. It implies that x^* in which $x^*_N = 0$ is cut out!

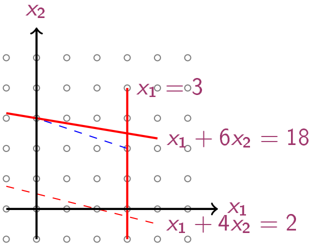
(theoretically it terminates after a finite number of iterations, but in practice not successful.)

Example

$$\begin{aligned} \max\ x_1 + 4x_2 & \\ x_1 + 6x_2 &\le 18 \\ x_1 \qquad &\le 3 \\ x_1, x_2 &\ge 0 \\ x_1, x_2 & \text{integer} \end{aligned}$$

	x1	x2	x3	x4	-z	b
	1	6	1	0	0	18
	1	0	0	1	0	3
	1	4	0	0	1	0

	x1	x2	x3	x4	-z	b
	0	1	1/6	-1/6	0	15/6
	1	0	0	1	0	3
	0	0	-2/3	-1/3	1	-13



$x_2 = 5/2, x_1 = 3$
Optimum, not integer

- We take the first row: | | 0 | 1 | 1/6 | -1/6 | 0 | 15/6 |
- CG cut $\sum_{j \in N} f_{uj} x_j \geq f_u \rightsquigarrow \frac{1}{6}x_3 + \frac{5}{6}x_4 \geq \frac{1}{2}$
- Let's verify that it is a CG cut:

$$
\begin{array}{l}
1/6\ (x_1 + 6x_2 \leq 18) \\
5/6\ (x_1 \qquad\quad \leq 3) \\
\hline
\quad x_1 + x_2 \leq 3 + 5/2 = 5.5
\end{array}
$$

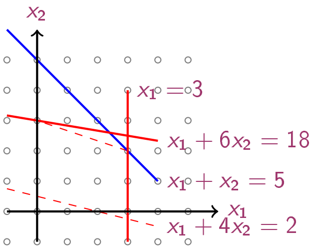
  since x_1, x_2 are integer $x_1 + x_2 \leq 5$. And it leaves out x^*.

- Let's see how it looks in the space of the original variables: from the first tableau:

$$
\begin{aligned}
x_3 &= 18 - 6x_2 - x_1 \\
x_4 &= 3 - x_1
\end{aligned}
$$

$$
\frac{1}{6}(18 - 6x_2 - x_1) + \frac{5}{6}(3 - x_1) \geq \frac{1}{2} \qquad \rightsquigarrow \qquad x_1 + x_2 \leq 5
$$

- Graphically:



- Let's continue:

	x1	x2	x3	x4	x5	-z	b
	0	0	-1/6	-5/6	1	0	-1/2
	0	1	1/6	-1/6	0	0	5/2
	1	0	0	1	0	0	3
	0	0	-2/3	-1/3	0	1	-13

We need to apply dual-simplex (will always be the case, why?)

ratio rule: $\min\{|\frac{c_j}{a_{ij}}| \; : \; a_{ij} < 0\}$

- After the dual simplex iteration:

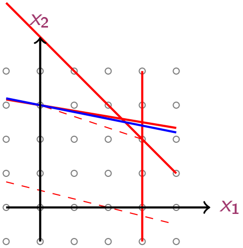
	x1	x2	x3	x4	x5	-z	b
	0	0	1/5	1	-6/5	0	3/5
	0	1	1/5	0	-1/5	0	13/5
	1	0	-1/5	0	6/5	0	12/5
	0	0	-3/5	0	-2/5	1	-64/5

We can choose any of the three rows.

Let's take the third: CG cut:
$\frac{4}{5}x_3 + \frac{1}{5}x_5 \geq \frac{2}{5}$

- In the space of the original variables:

$$4(18 - x_1 - 6x_2) + (5 - x_1 - x_2) \geq 2$$
$$x_1 + 5x_2 \leq 15$$



- ...