DM811
HEURISTICS AND LOCAL SEARCH ALGORITHMS
FOR COMBINATORIAL OPTIMZATION

Lecture 12

Stochastic Local Search Methods

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Outline

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1. Stochastic Local Search Methods (Metaheuristics)
   Randomized Iterative Improvement
   Attribute Based Hill Climber
   Dynamic Local Search
   Iterated Local Search
   Tabu Search

‘Simple’ SLS Methods

Goal:

Effectively escape from local minima of given evaluation function.

General approach:

For fixed neighborhood, use step function that permits
worsening search steps.

Specific methods:

- Randomized Iterative Improvement
- (Simulated Annealing)
- Attribute Based Hill Climber
- Dynamic Local Search
- Iterated Local Search
- Tabu Search

Min-Conflict Heuristics

procedure *WalkSAT* (*F*, *maxTries*, *maxSteps*, *slc*)
  input: *CNF formula F, positive integers maxTries and maxSteps, heuristic function slc*
  output: *model of F* **or** 'no solution found'
  for *try* := 1 **to** *maxTries* **do**
    a := randomly chosen assignment of the variables in formula *F*;
    for *step* := 1 **to** *maxSteps* **do**
      if *a* satisfies *F* **then return** *a* **end**
      c := randomly selected clause unsatisfied under *a*;
      x := variable selected from *c* according to heuristic function *slc*;
      a := *a* with *x* flipped;
    end
  end
  return 'no solution found'
end *WalkSAT*

Figure 6.3 The WalkSAT algorithm family. All random selections are according to a uniform probability distribution over the underlying sets; WalkSAT algorithms differ in the variable selection heuristic *slc*.

Randomized Iterative Improvement

Key idea: In each search step, with a fixed probability perform an uninformed random walk step instead of an iterative improvement step.

Randomized Iterative Improvement (RII):
determine initial candidate solution s
while termination condition is not satisfied **do**
  With probability *wp*:
    choose a neighbor s' of s uniformly at random
  Otherwise:
    choose a neighbor s' of s such that $g(s') < g(s)$ or,
     if no such s' exists, choose s' such that $g(s')$ is minimal
  $s := s'$

Example: Randomized Iterative Improvement for GCP

procedure *RIIGCP*$(F, wp, maxSteps)$
  input: *a graph* G *and* k, *probability wp, integer maxSteps*
  output: *a proper coloring* φ *for* G **or** $\emptyset$
  choose coloring φ of G uniformly at random;
  steps := 0;
  while not(φ is not proper) **and** (*steps* < *maxSteps*) **do**
    with probability *wp* **do**
      select v in V and c in Γ uniformly at random;
    otherwise
      select v in V^c and c in Γ uniformly at random from those that
        maximally decrease number of edge violations;
    change color of v in φ;
    steps := *steps*+1;
  end
  if φ is proper for G **then return** φ
  else return $\emptyset$
  end
end *RIIGCP*

Note:

- No need to terminate search when local minimum is encountered

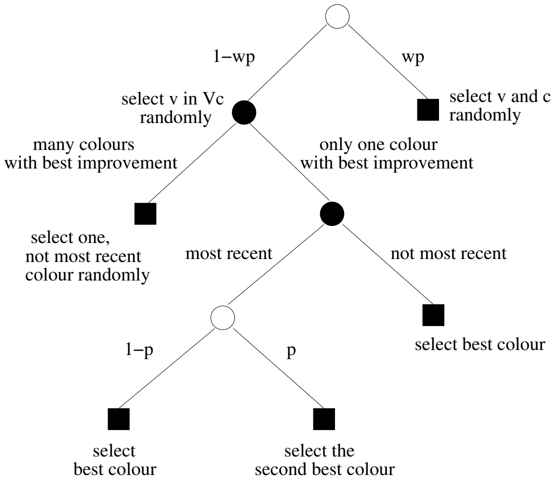
  Instead: Impose limit on number of search steps or CPU time, from beginning of search or after last improvement.
- Probabilistic mechanism permits arbitrary long sequences of random walk steps

  Therefore: When run sufficiently long, RII is guaranteed to find (optimal) solution to any problem instance with arbitrarily high probability.
- A variant of RII has successfully been applied to SAT (GWSAT algorithm)
- A variant of GUWSAT, GWSAT [Selman et al., 1994], was at some point state-of-the-art for SAT.
- Generally, RII is often outperformed by more complex LS methods.

Novelty

Key idea: combine Randomized Iterative Improvement with Min-Conflicts

Example on GCP



Attribute Based Hill Climber

- attributes are solution elements that change in a move
- each attribute has associated a value:
  - the value of the best solution visited that contains it
  - infinity, otherwise
- at each step, a solution in N is acceptable iff it contains an attribute that has never been seen in a solution of such high quality before

$$N'(s) = \{s' \in N(s) \ : \ \exists a \in s' \text{ s.t. } f(s') < \phi(a)\}$$

where

$$\phi(a) = \begin{cases} \infty & \text{if } V \cap S^a = \emptyset \\ \min\{f(s) : s \in V \cap S^a\} & \text{otherwise} \end{cases}$$

with V set of visited solutions and S^a set of solutions that contains a.

Examples

- TSP:

$$A_{TSP} = \{(i,j) \mid (i,j) \in E\}$$

- QAP:

$$A_{QAP} = \{(i,j) \mid 1 \le i \le n, 1 \le j \le n\} \text{ representing } (\varphi(i), j)$$

ALGORITHM 1. accepts(s, t)

if $\exists a \in (t - s)$ s.t. $f(t) <$ amem$[a]$ **then**
  return true
else if $f(t) \geqslant f(s)$ **then**
  return false
else
  for $i = 1$ to $|s - t| + 1$ **do**
    $a \leftarrow$ worst(i, s)
    if $f(t) \geqslant$ amem$[a]$ **then**
      return false
    else if $a \notin (s - t)$ **then**
      return true
    end if
  end for
end if

Dynamic Local Search

- **Key Idea:** Modify the evaluation function whenever a local optimum is encountered.
- Associate *weights* (*penalties*) with solution components; these determine impact of components on evaluation function value.
- Perform Iterative Improvement; when in local minimum, increase penalties of some solution components until improving steps become available.

Dynamic Local Search (DLS):
determine *initial candidate solution* s
initialize penalties
while *termination criterion* is not satisfied **do**
  compute *modified evaluation function* g' from g
    based on *penalties*
  perform *subsidiary local search* on s
    using *evaluation function* g'
  update penalties based on s

Dynamic Local Search (continued)

- **Modified evaluation function:**

$$g'(\pi, s) := g(\pi, s) + \sum_{i \in SC(\pi', s)} \mathtt{penalty}(i),$$

  where $SC(\pi', s)$ is the set of solution components of problem instance π' used in candidate solution s.

- **Penalty initialization:** For all i: $\mathtt{penalty}(i) := 0$.
- **Penalty update** in local minimum s: Typically involves *penalty increase* of some or all solution components of s; often also occasional *penalty decrease* or *penalty smoothing*.
- **Subsidiary local search:** Often *Iterative Improvement*.

Potential problem:

Solution components required for (optimal) solution may also be present in many local minima.

Possible solutions:

A: Occasional decreases/smoothing of penalties.

B: Only increase penalties of solution components that are least likely to occur in (optimal) solutions.

Implementation of B:

Only increase penalties of solution components i with maximal utility [Voudouris and Tsang, 1995]:

$$\mathtt{util}(s', i) := \frac{g_i(\pi, s')}{1 + \mathtt{penalty}(i)}$$

where $g_i(\pi, s')$ is the solution quality contribution of i in s'.

Example: Guided Local Search (GLS) for the TSP

[Voudouris and Tsang 1995; 1999]

- **Given:** TSP instance G
- **Search space:** Hamiltonian cycles in G with n vertices;
- **Neighborhood:** 2-edge-exchange;
- **Solution components** edges of G;
  $g_e(G, p) := w(e)$;
- **Penalty initialization:** Set all edge penalties to zero.
- **Subsidiary local search:** Iterative First Improvement.
- **Penalty update:** Increment penalties for all edges with maximal utility by

$$\lambda := 0.3 \cdot \frac{w(s_{\textit{2-opt}})}{n}$$

  where $s_{\textit{2-opt}}$ = 2-optimal tour.

Hybrid Methods

Combination of 'simple' methods often yields
substantial performance improvements.

Simple examples:

- Commonly used restart mechanisms can be seen as hybridisations with Uninformed Random Picking
- Iterative Improvement + Uninformed Random Walk = Randomized Iterative Improvement

Iterated Local Search

Key Idea: Use two types of LS steps:

- *subsidiary local (local) search* steps for reaching local optima as efficiently as possible (intensification)
- *perturbation steps* for effectively escaping from local optima (diversification).

Also: Use *acceptance criterion* to control diversification *vs* intensification behavior.

Iterated Local Search (ILS):
determine initial candidate solution s
perform *subsidiary local search* on s
while termination criterion is not satisfied **do**
  $r := s$
  perform *perturbation* on s
  perform *subsidiary local search* on s
  based on *acceptance criterion*,
    keep s or revert to $s := r$

Note:

- *Subsidiary local search* results in a local minimum.
- ILS trajectories can be seen as walks in the space of local minima of the given evaluation function.
- *Perturbation phase* and *acceptance criterion* may use aspects of *search history* (*i.e.*, limited memory).
- In a high-performance ILS algorithm, *subsidiary local search*, *perturbation mechanism* and *acceptance criterion* need to complement each other well.

Subsidiary local search:

- More effective subsidiary local search procedures lead to better ILS performance.
  Example: 2-opt *vs* 3-opt *vs* LK for TSP.
- Often, subsidiary local search = iterative improvement, but more sophisticated LS methods can be used. (*e.g.*, Tabu Search).

Perturbation mechanism:

- Needs to be chosen such that its effect *cannot* be easily undone by subsequent local search phase.
  (Often achieved by search steps larger neighborhood.)
  Example: local search = 3-opt, perturbation = 4-exchange steps in ILS for TSP.
- A perturbation phase may consist of one or more perturbation steps.
- Weak perturbation ⇒ short subsequent local search phase; *but:* risk of revisiting current local minimum.
- Strong perturbation ⇒ more effective escape from local minima; *but:* may have similar drawbacks as random restart.
- Advanced ILS algorithms may change nature and/or strength of perturbation adaptively during search.

Acceptance criteria:

- Always accept the best of the two candidate solutions

  ⇒ ILS performs Iterative Improvement in the space of local optima reached by subsidiary local search.

- Always accept the most recent of the two candidate solutions

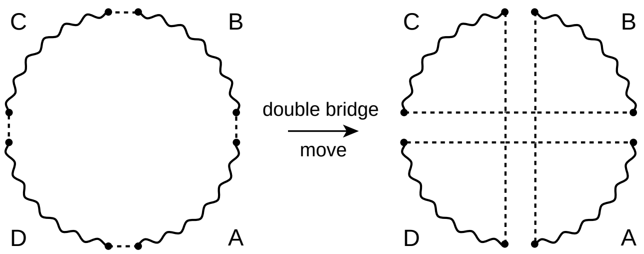
  ⇒ ILS performs random walk in the space of local optima reached by subsidiary local search.

- Intermediate behavior: select between the two candidate solutions based on the *Metropolis criterion* (*e.g.*, used in *Large Step Markov Chains* [Martin *et al.*, 1991].

- Advanced acceptance criteria take into account search history, *e.g.*, by occasionally reverting to *incumbent solution*.

Example: Iterated Local Search for the TSP (1)

- **Given:** TSP instance G.
- **Search space:** Hamiltonian cycles in G.
- **Subsidiary local search:** Lin-Kernighan variable depth search algorithm
- **Perturbation mechanism:**
  'double-bridge move' = particular 4-exchange step:



- **Acceptance criterion:** Always return the best of the two given candidate round trips.

Example: Iterated Local Search for the TSP (2)

Note:

- Double-bridge move local cannot be directly reversed by a sequence of 2-exchange steps as performed by "usual" LK implementations.
- This perturbation is empirically shown to be effective independent of instance size.

Note:

- This ILS algorithm for the TSP is known as *Iterated Lin-Kernighan (ILK) Algorithm*.

- Although ILK is structurally rather simple, an efficient implementation was shown to achieve excellent performance [Johnson and McGeoch, 1997].

Iterated local search algorithms ...

- are typically rather easy to implement (given existing implementation of subsidiary simple LS algorithms);
- achieve state-of-the-art performance on many combinatorial problems, including the TSP.

There are many LS approaches that are closely related to ILS, including:

- Large Step Markov Chains [Martin *et al.*, 1991]
- Chained Local Search [Martin and Otto, 1996]
- Variants of Variable Neighbourhood Search (VNS) [Hansen and Mladenovič, 2002]

Tabu Search

Key idea: Use aspects of search history (memory) to escape from local minima.

- Associate *tabu attributes* with candidate solutions or solution components.
- Forbid steps to search positions recently visited by underlying iterative best improvement procedure based on tabu attributes.

Tabu Search (TS):

determine initial candidate solution s
While *termination criterion* is not satisfied:
| *determine set N' of non-tabu neighbors of s*
| *choose a best candidate solution s' in N'*
|
| *update tabu attributes* based on s'
| $s := s'$

Example: Tabu Search for GCP – TabuCol

- **Search space:** set of all complete colorings of G.
- **Solution set:** proper colorings of G.
- **Neighborhood relation:** one-exchange.
- **Memory:** Associate tabu status (Boolean value) with each pair (v, c).
- **Initialization:** a construction heuristic
- **Search steps:**
  - pairs (v, c) are tabu if they have been changed in the last *tt* steps;
  - neighboring colorings are admissible if they can be reached by changing a non-tabu pair or have fewer unsatisfied edge constr. than the best coloring seen so far (*aspiration criterion*);
  - choose uniformly at random admissible coloring with minimal number of unsatisfied constraints.
- **Termination:** upon finding a proper coloring for G *or* after given bound on number of search steps has been reached *or* after a number of idle iterations

Note:

- Non-tabu search positions in $N(s)$ are called *admissible neighbors of s*.
- After a search step, the current search position or the solution components just added/removed from it are declared *tabu* for a fixed number of subsequent search steps (*tabu tenure*).
- Often, an additional *aspiration criterion* is used: this specifies conditions under which tabu status may be overridden (*e.g.*, if considered step leads to improvement in incumbent solution).
- Crucial for efficient implementation:
  - keep time complexity of search steps minimal by using special data structures, incremental updating and caching mechanism for evaluation function values;
  - efficient determination of tabu status: store for each variable x the number of the search step when its value was last changed it_x; x is tabu if $it - it_x < tt$, where it = current search step number.

Note: Performance of Tabu Search depends crucially on setting of tabu tenure *tt*:

- *tt* too low $\Rightarrow$ search stagnates due to inability to escape from local minima;
- *tt* too high $\Rightarrow$ search becomes ineffective due to overly restricted search path (admissible neighborhoods too small)

Advanced TS methods:

- **Robust Tabu Search** [Taillard, 1991]:
  repeatedly choose *tt* from given interval;
  also: force specific steps that have not been made for a long time.
- **Reactive Tabu Search** [Battiti and Tecchiolli, 1994]:
  dynamically adjust *tt* during search;
  also: use escape mechanism to overcome stagnation.

Further improvements can be achieved by using *intermediate-term* or *long-term memory* to achieve additional *intensification* or *diversification*.

Examples:

- Occasionally backtrack to *elite candidate solutions*, *i.e.*, high-quality search positions encountered earlier in the search; when doing this, all associated tabu attributes are cleared.
- Freeze certain solution components and keep them fixed for long periods of the search.
- Occasionally force rarely used solution components to be introduced into current candidate solution.
- Extend evaluation function to capture frequency of use of candidate solutions or solution components.

Tabu search algorithms algorithms are state of the art for solving many combinatorial problems, including:

- SAT and MAX-SAT
- the Constraint Satisfaction Problem (CSP)
- many scheduling problems

Crucial factors in many applications:

- choice of neighborhood relation
- efficient evaluation of candidate solutions
  (caching and incremental updating mechanisms)

Example: Tabu Search for QAP

- **Solution representation**: permutation π
- **Initial Solution**: randomly generated
- **Neighborhood**: interchange
  $\Delta_I : \quad \delta(\pi) = \{\pi' | \pi'_k = \pi_k \text{ for all } k \neq \{i,j\} \text{ and } \pi'_i = \pi_j, \pi'_j = \pi_i\}$
- **Tabu status**: forbid δ that place back the items in the positions they have already occupied in the last *tt* iterations (short term memory)
- Implementation details:
  - compute $g(\pi') - f(\pi)$ in $O(n)$ or $O(1)$ by storing the values all possible previous moves.
  - maintain a matrix $[T_{ij}]$ of size $n \times n$ and write the last time item i was moved in location k plus tt
  - δ is tabu if it satisfies both:
    - $T_{i,\pi(j)} \geq$ current iteration
    - $T_{j,\pi(i)} \geq$ current iteration

Example: Robust Tabu Search for QAP

- **Aspiration criteria**:
  - allow forbidden δ if it improves the last π^*
  - select δ if never chosen in the last A iterations (long term memory)
- Parameters: $tt \in [\lfloor 0.9n \rfloor, \lceil 1.1n + 4 \rceil]$ and $A = 5n^2$

Example: Reactive Tabu Search for QAP

- **Aspiration criteria**:
  - allow forbidden δ if it improves the last π^*
- **Tabu Tenure**
  - maintain a hash table (or function) to record previously visited solutions
  - increase *tt* by a factor $\alpha_{inc}(= 1.1)$ if the current solution was previously visited
  - decrease *tt* by a factor $\alpha_{dec}(= 0.9)$ if *tt* not changed in the last $sttc$ iterations or all moves are tabu
- Trigger escape mechanism if a solution is visited more than $nr(= 3)$ times
- Escape mechanism $= 1 + (1 + r) \cdot ma/2$ random moves