DMP204
SCHEDULING,
TIMETABLING AND ROUTING

Lecture 13

Parallel Machine Models

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Outline

1. Parallel Machine Models

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$Pm \,|\,|\, C_{max}$ (without preemption)

$Pm \,|\,|\, C_{max}$ LPT heuristic, approximation ratio: $\frac{4}{3} - \frac{1}{3m}$

$P\infty \,|\, prec \,|\, C_{max}$ CPM

$Pm \,|\, prec \,|\, C_{max}$ strongly NP-hard, LNS heuristic (non optimal)

$Pm \,|\, p_j = 1, M_j \,|\, C_{max}$ LFJ-LFM (optimal if M_j are nested)

$Pm \mid prmp \mid C_{max}$ (with preemption)

Not NP-hard:

- Linear Programming (exercise)
- Construction based on $LWB = \max\left\{p_1, \sum_{j=1}^{n} \frac{p_j}{m}\right\}$
- Dispatching rule: longest remaining processing time (LRPT) optimal in discrete time

Decomposition

If model has "block" structure

$$\begin{array}{llllll} \max & c^1x^1 & + & c^2x^2 & + \dots + & c^Kx^K & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \dots + & A^Kx^K & = b \\ & D^1x^1 & + & & & & \leq d_1 \\ & & + & D^2x^2 & & & \leq d_2 \\ & & & & \dots & & \leq \vdots \\ & & & & & D^Kx^K & \leq d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \dots & x^K \in \mathbb{Z}_+^{n_K} & \end{array}$$

Lagrangian relaxation

Objective becomes

$$c^1x^1 + c^2x^2 + \dots + c^Kx^K$$
$$-\lambda\left(A^1x^1 + A^2x^2 + \dots + A^Kx^K - b\right)$$

Decomposed into

$$\begin{array}{llllll} \max & c^1x^1 - \lambda A^1x^1 + c^2x^2 - \lambda A^2x^2 + \dots + c^Kx^K - \lambda A^Kx^K + b & & & & \\ \text{s.t.} & D^1x^1 & + & & & \leq d_1 \\ & & + & D^2x^2 & & \leq d_2 \\ & & & \dots & & \leq \vdots \\ & & & & D^Kx^K & \leq d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \dots \quad x^K \in \mathbb{Z}_+^{n_K} & \end{array}$$

Model is separable

Dantzig-Wolfe decomposition

If model has "block" structure

$$\begin{array}{llllll} \max & c^1x^1 & + & c^2x^2 & + \dots + & c^Kx^K & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \dots + & A^Kx^K & = b \\ & D^1x^1 & + & & & & \leq d_1 \\ & & + & D^2x^2 & & & \leq d_2 \\ & & & & \dots & & \leq \vdots \\ & & & & & D^Kx^K & \leq d_K \\ & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \dots & x^K \in \mathbb{Z}_+^{n_K} & \end{array}$$

Substituting each set X^k, $k = 1, \dots, K$ in original model getting *Master Problem*

$$\max c^1\Big(\sum_{t\in T_1} \lambda_{1,t} x^{1,t}\Big) + c^2\Big(\sum_{t\in T_2} \lambda_{2,t} x^{2,t}\Big) + \dots + c^K\Big(\sum_{t\in T_K} \lambda_{K,t} x^{K,t}\Big)$$

$$\text{s.t.} \quad A^1\Big(\sum_{t\in T_1} \lambda_{1,t} x^{1,t}\Big) + A^2\Big(\sum_{t\in T_2} \lambda_{2,t} x^{2,t}\Big) + \dots + A^K\Big(\sum_{t\in T_K} \lambda_{K,t} x^{K,t}\Big) = b$$

$$\sum_{t\in T_k} \lambda_{k,t} = 1 \qquad k = 1, \dots, K$$

$$\lambda_{k,t} \in \{0,1\}, \qquad t \in T_k \quad k = 1, \dots, K$$

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$\begin{array}{llllll} \max & c^1x^1 & + & c^2x^2 & + \dots + & c^kx^k & \\ \text{s.t.} & A^1x^1 & + & A^2x^2 & + \dots + & A^kx^k & = b \\ & x^1 \in \text{conv}(X^1) & & x^2 \in \text{conv}(X^2) & \dots & x^k \in \text{conv}(X^k) & \end{array}$$

Strength of Lagrangian relaxation

- z^{LPM} be LP-solution value of master problem
- z^{LD} be solution value of lagrangian dual problem

(Theorem 11.2)

$$z^{LPM} = z^{LD}$$

Delayed column generation, linear master

(*minimization problem*)

Run Simplex algorithm as if complete master problem was known

- Start with a basis solution
- Solve
  $$x_B = A_B^{-1}b$$
  and find dual variables
  $$y = c_B A_B^{-1}$$
- When choosing entering variable solve pricing problem which minimizes reduced costs
  $$c_j^r = c_j - yA_j$$
- If $c_j^r < 0$ add corresponding column A_j to model and repeat
- If $c_j^r \geq 0$ stop

Cutting Stock Problem

(*minimization problem*)

a_{ij} is number of pieces of type i cut from pattern j
Master problem,

$$\begin{array}{ll} \min & \sum_{j=1}^n u_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij}u_j \geq b_i \quad i = 1,\ldots,m, j = 1,\ldots,n \\ & x_{ij} \in \mathbb{Z}_+ \end{array}$$

Solving linear master through delayed column generation

- Start with patterns which only contain one type i
- Solve restricted master
- Dual variables y_i say how "attractive" a type i is
- Pricing problem
  $$z^S = \min \ 1 - \sum_{i=1}^m y_i x_i$$
  $$\text{s.t.} \ \sum_{i=1}^m w_i x_i \leq L$$
  $$x \geq 0, \text{integer}$$
- stop if $z^S \geq 0$

Terminology

- Master Problem
- Restricted Master Problem
- Subproblem or Pricing Problem
- Branch and cut:
  Branch-and-bound algorithm using cuts to strengthen bounds.
- Branch and price:
  Branch-and-bound algorithm using column generation to derive bounds.
- One says that discarded columns are "priced out".

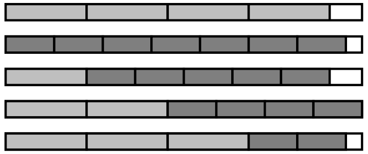
Branch-and-price

- LP-solution of master problem may have fractional solutions
- Branch-and-bound for getting IP-solution
- In each node solve LP-relaxation of master
- Subproblem may change when we add constraints to master problem
- Branching strategy should make subproblem easy to solve

Branch-and-price, example

The matrix A contains all different cutting patterns

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$



Problem

$$\begin{aligned} \text{minimize } & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to } & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{aligned}$$

LP-solution $\lambda_1 = 1.375, \lambda_4 = 0.75$

Branch on $\lambda_1 = 0$, $\lambda_1 = 1$, $\lambda_1 = 2$

- Column generation may not generate pattern (4,0)
- Pricing problem is knapsack problem with pattern forbidden

Tailing off effect

Column generation may converge slowly in the end

- We do not need exact solution, just lower bound
- Solving master problem for subset of columns does not give valid lower bound (why?)
- Instead we may use Lagrangian relaxation of joint constraint
- “guess” lagrangian multipliers equal to dual variables from master problem

Heuristic solution

- Restricted master problem will only contain a subset of the columns
- We may solve restricted master problem to IP-optimality
- Restricted master is a “set-covering-like” problem which is not too difficult to solve