DMP204
SCHEDULING,
TIMETABLING AND ROUTING

Lecture 14
Flow Shop Models

Marco Chiarandini

Outline

1. Parallel Machine Models

2. Flow Shop
   - Introduction
   - Makespan calculation
   - Johnson's algorithm
   - Construction heuristics
   - Iterated Greedy
   - Efficient Local Search and Tabu Search

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Identical machines

Min makespan, without preemption

$Pm \,|\,|\, C_{max}$: LPT heuristic, approximation ratio: $\frac{4}{3} - \frac{1}{3m}$

$P\infty \,|\, prec \,|\, C_{max}$: CPM

$Pm \,|\, prec \,|\, C_{max}$: strongly NP-hard, LNS heuristic (non optimal)

$Pm \,|\, p_j = 1, M_j \,|\, C_{max}$: least flexible job (LFJ) - least flexible machine (LFM) (optimal if M_j are nested)

Identical machines

Min makespan, with preemption

$Pm \,|\,|\, C_{max}$: Not NP-hard:

- Linear Programming (exercise)

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Identical machines

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- Linear Programming (exercise)
- Construction based on $LWB = \max\left\{p_1, \sum_{j=1}^{n} \frac{p_j}{m}\right\}$

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- Linear Programming (exercise)
- Construction based on $LWB = \max\left\{p_1, \sum_{j=1}^{n} \frac{p_j}{m}\right\}$
- Dispatching rule: longest remaining processing time (LRPT)
  optimal in discrete time

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Uniform machines

$Qm \,|\, prmp \,|\, C_{max}$

- Construction based on

$$LWB = \max\left\{\frac{p_1}{v_1}, \frac{p_1 + p_2}{v_1 + v_2}, \dots, \frac{\sum_{j=1}^{n} p_j}{\sum_{j=1}^{m} v_j}\right\}$$

- Dispatching rule: longest remaining processing time on the fastest machine first (processor sharing)
  optimal in discrete time

Unrelated machines

$R||\sum_j C_j$ is NP-hard

Solved by local search methods.

- Solution representation

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  - a collection of m sequences, one for each job

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    recall that $1||\sum w_j C_j$ is solvable in $O(n \log n)$

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  - indirect representation
    assignment of jobs to machines
    the sequencing is left to the optimal SWPT rule

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- Neighborhood: one exchange, swap
- Evaluation function. How costly is the computation?

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Flow Shop

General Shop Scheduling:

- $J = \{1, \ldots, N\}$ set of jobs; $M = \{1, 2, \ldots, m\}$ set of machines

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Flow Shop Scheduling:

- $\mu_{ij} = l$, $l = 1, 2, \ldots, m$
- precedence constraints: $O_{ij} \rightarrow O_{i+1,j}$, $i = 1, 2, \ldots, n$ for all jobs

Example

$jobs$	j_1	j_2	j_3	j_4	j_5
p_{1,j_k}	5	5	3	6	3
p_{2,j_k}	4	4	2	4	4
p_{3,j_k}	4	4	3	4	1
p_{4,j_k}	3	6	3	2	5

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schedule representation
$\pi_1, \pi_2, \pi_3, \pi_4$:

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$\pi_1, \pi_2, \pi_3, \pi_4$:

$\pi_1 : O_{11}, O_{12}, O_{13}, O_{14}$

$\pi_2 : O_{21}, O_{22}, O_{23}, O_{24}$

$\pi_3 : O_{31}, O_{32}, O_{33}, O_{34}$

$\pi_4 : O_{41}, O_{42}, O_{43}, O_{44}$

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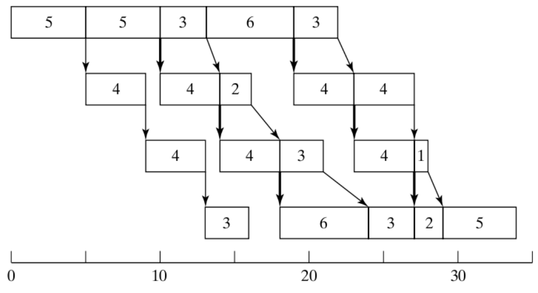
Gantt chart



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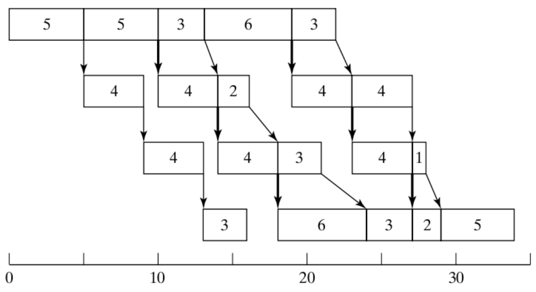
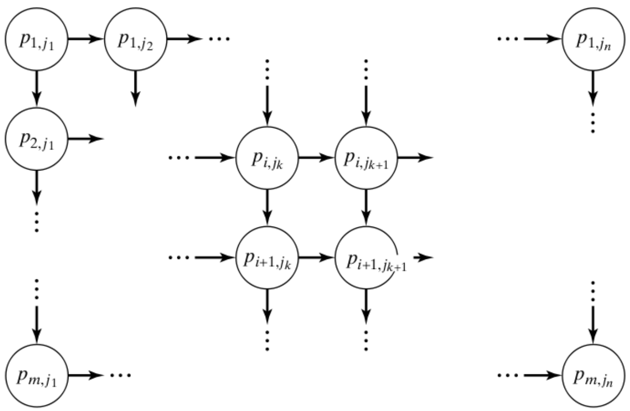


- we assume unlimited buffer
- if same job sequence on each machine ➨ permutation flow shop

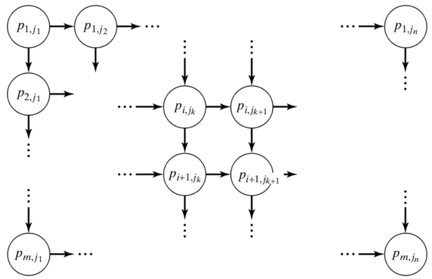
Outline

Directed Graph Representation

Given a sequence: operation-on-node network,
jobs on columns, and machines on rows

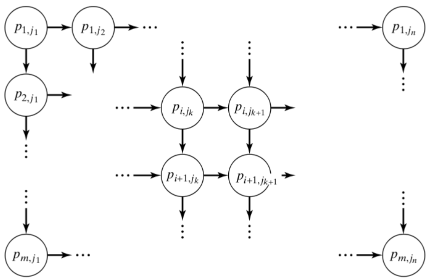


Directed Graph Representation



Recursion for C_{max}

Directed Graph Representation



Recursion for C_{max}

$$C_{i,\pi(1)} = \sum_{l=1}^{i} p_{l,\pi(1)}$$

$$C_{1,\pi(j)} = \sum_{l=1}^{j} p_{l,\pi(l)}$$

$$C_{i,\pi(j)} = \max\{C_{i-1,\pi(j)}, C_{i,\pi(j-1)}\} + p_{i,\pi(j)}$$

Computation cost?

Example

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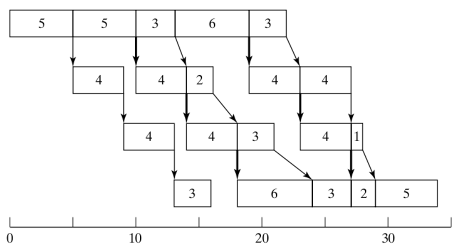
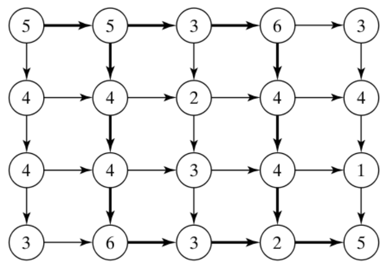
$C_{max} = 34$

corresponds to longest path





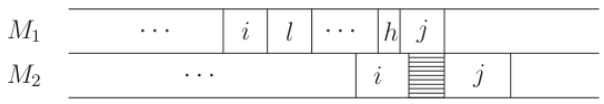
Outline

$Fm\,|\,|\,C_{max}$

Theorem

There always exist an optimum sequence without change in the first two and last two machines.

Proof: By contradiction.



Corollary

$F2\,|\,|\,C_{max}$ and $F3\,|\,|\,C_{max}$ are permutation flow shop

Note: $F3\,|\,|\,C_{max}$ is strongly NP-hard

$F2\,|\,|\,C_{max}$

Intuition:

$F2\,|\,|\,C_{max}$

Intuition: give something short to process to 1 such that 2 becomes operative and give something long to process to 2 such that its buffer has time to fill.

$F2 \mid \mid C_{max}$

Intuition: give something short to process to 1 such that 2 becomes operative and give something long to process to 2 such that its buffer has time to fill.

Constructs a sequence $T : T(1), \ldots, T(n)$ to process in the same order on both machines by concatenating two sequences:
a left sequence $L : L(1), \ldots, L(t)$, and a right sequence $R : R(t+1), \ldots, R(n)$, that is, $T = L \circ R$

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[Selmer Johnson, 1954, Naval Research Logistic Quarterly]

Let J be the set of jobs to process
Let $T, L, R = \emptyset$

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Let J be the set of jobs to process
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Step 1 Find (i^*, j^*) such that $p_{i^*,j^*} = \min\{p_{ij} \mid i \in 1, 2, j \in J\}$

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Step 2 If $i^* = 1$ then $L = L \circ \{i^*\}$
else if $i^* = 2$ then $R = R \circ \{i^*\}$

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$F2\,|\,|\,C_{max}$

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Step 2 If $i^* = 1$ then $L = L \circ \{i^*\}$
else if $i^* = 2$ then $R = R \circ \{i^*\}$
Step 3 $J := J \setminus \{j^*\}$
Step 4 If $J \neq \emptyset$ go to Step 1 else $T = L \circ R$

Theorem
The sequence $T : T(1), , \dots, T(n)$ is optimal.

Proof

- Assume at one iteration of the algorithm that job k has the min processing time on machine 1. Show that in this case job k has to go first on machine 1 than any other job selected later.
- By contradiction, show that if in a schedule S a job j precedes k on machine 1 and has larger processing time on 1, then S is a worse schedule than S'.
  There are three cases to consider.
- Iterate the prove for all jobs in L.
- Prove symmetrically for all jobs in R.

$Fm \,|\, prmu, p_{ij} = p_j \,|\, C_{max}$

[Proportionate permutation flow shop]

- **Theorem:** $C_{max} = \sum_{j=1}^{n} p_j + (m-1)\max(p_1, \dots, p_n)$ and is sequence independent
- Generalization to include machines with different speed: $p_{ij} = p_j / v_i$

  Theorem:
  if the first machine is the bottleneck then LPT is optimal.
  if the last machine is the bottleneck then SPT is optimal.

Outline

Construction Heuristics (1)

$Fm \mid prmu \mid C_{max}$

Slope heuristic

- schedule in decreasing order of $A_j = -\sum_{i=1}^{m}(m-(2i-1))p_{ij}$

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Construction Heuristics (1)
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Slope heuristic
- schedule in decreasing order of $A_j = -\sum_{i=1}^{m}(m-(2i-1))p_{ij}$

Campbell, Dudek and Smith's heuristic (1970)

extension of Johnson's rule to when permutation is not dominant

- recursively create 2 machines 1 and $m-1$

$$p'_{ij} = \sum_{k=1}^{i} p_{kj} \qquad p''_{ij} = \sum_{k=m-i+1}^{m} p_{kj}$$

  and use Johnson's rule
- repeat for all $m-1$ possible pairings
- return the best for the overall m machine problem

Construction Heuristics (2)

$Fm \mid prmu \mid C_{max}$

Nawasz, Enscore, Ham's heuristic (1983)

Step 1: order in decreasing $\sum_{j=1}^{m} p_{ij}$

Step 2: schedule the first 2 jobs at best

Step 3: insert all others in best position

Implementation in $O(n^2m)$

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Implementation in $O(n^2m)$

[Framinan, Gupta, Leisten (2004)] examined 177 different arrangements of jobs in Step 1 and concluded that the NEH arrangement is the best one for C_{max}.

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Iterated Greedy

$Fm \mid prmu \mid C_{max}$

Iterated Greedy [Ruiz, Stützle, 2007]

Destruction: remove d jobs at random

Construction: reinsert them with NEH heuristic in the order of removal

Local Search: insertion neighborhood
(first improvement, whole evaluation $O(n^2m)$)

Acceptance Criterion: random walk, best, SA-like

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Iterated Greedy [Ruiz, Stützle, 2007]

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Performance on up to $n = 500 \times m = 20$:

- NEH average gap 3.35% in less than 1 sec.
- IG average gap 0.44% in about 360 sec.

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Efficient local search for $Fm \,|\, prmu \,|\, C_{max}$

Tabu search (TS) with insert neighborhood.

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Tabu search (TS) with insert neighborhood.

TS uses best strategy. ➨ need to search efficiently!

Neighborhood pruning [Novicki, Smutnicki, 1994, Grabowski, Wodecki, 2004]

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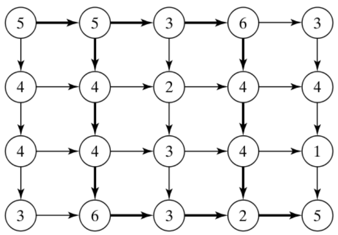

A sequence $t = (t_1, t_2, \ldots, t_{m-1})$ defines a path in π:

$$C(\pi, t) = \sum_{j=1}^{t_1} p_{\pi(j)1} + \sum_{j=t_1}^{t_2} p_{\pi(j)2} + \cdots + \sum_{j=t_{m-1}}^{n} p_{\pi(j)m}.$$

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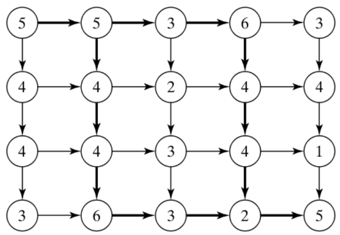
TS uses best strategy. ➨ need to search efficiently!

Neighborhood pruning [Novicki, Smutnicki, 1994, Grabowski, Wodecki, 2004]



A sequence $t = (t_1, t_2, \ldots, t_{m-1})$ defines a path in π:

$$C(\pi, t) = \sum_{j=1}^{t_1} p_{\pi(j)1} + \sum_{j=t_1}^{t_2} p_{\pi(j)2} + \cdots + \sum_{j=t_{m-1}}^{n} p_{\pi(j)m}.$$

C_{max} expression through critical path:

$$C_{\max}(\pi) = \max_{1 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{m-1} \leqslant n} \left(\sum_{j=1}^{t_1} p_{\pi(j)1} + \sum_{j=t_1}^{t_2} p_{\pi(j)2} + \cdots + \sum_{j=t_{m-1}}^{n} p_{\pi(j)m} \right)$$

critical path: $\vec{u} = (u_1, u_2, \ldots, u_m) : C_{max}(\pi) = C(\pi, u)$

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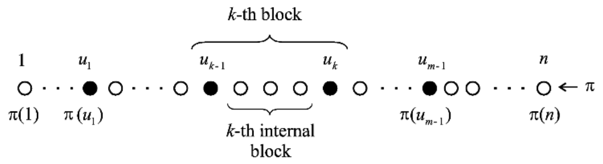
Block B_k and Internal Block B_k^{Int}



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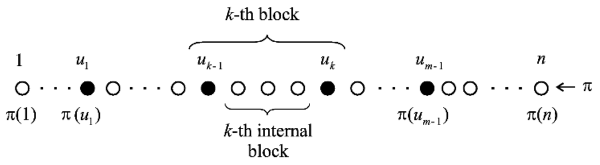

Theorem (Werner, 1992)

Let $\pi, \pi' \in \Pi$, if π' has been obtained from π by an job insert so that $C_{max}(\pi') < C_{max}(\pi)$ then in π':

a) *at least one job $j \in B_k$ precedes job $\pi(u_{k-1})$, $k = 1, \ldots, m$*

b) *at least one job $j \in B_k$ succeeds job $\pi(u_k)$, $k = 1, \ldots, m$*

Corollary (Elimination Criterion)

If π' is obtained by π by an "internal block insertion" then $C_{max}(\pi') \geq C_{max}(\pi)$.

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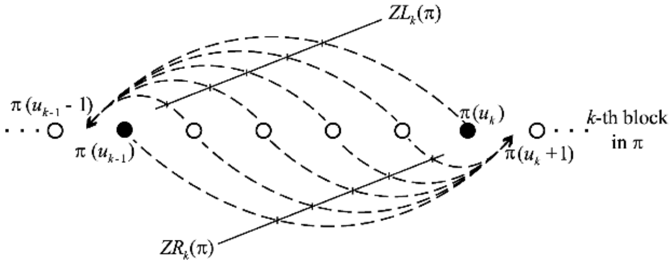
Hence we can restrict the search to where the good moves can be:



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Further speedup: Use of lower bounds in delta evaluations:
Let δ^r_{x,u_k} indicate insertion of x after u_k (move of type $ZR_k(\pi)$)

$$\Delta(\delta^r_{x,u_k}) = \begin{cases} p_{\pi(x),k+1} - p_{\pi(u_k),k+1} & x \neq u_{k-1} \\ p_{\pi(x),k+1} - p_{\pi(u_k),k+1} + p_{\pi(u_{k-1}+1),k-1} - p_{\pi(x),k-1} & x = u_{k-1} \end{cases}$$

That is, add and remove from the adjacent blocks

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That is, add and remove from the adjacent blocks
It can be shown that:

$$C_{max}(\delta^r_{x,u_k}(\pi)) \geq C_{max}(\pi) + \Delta(\delta^r_{x,u_k})$$

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Theorem (Nowicki and Smutnicki, 1996, EJOR)

The neighborhood thus defined is connected.

Metaheuristic details:

Prohibition criterion:
an insertion δ_{x,u_k} is tabu if it restores the relative order of $\pi(x)$ and $\pi(x+1)$.

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Tabu length: $TL = 6 + \left[\frac{n}{10m}\right]$

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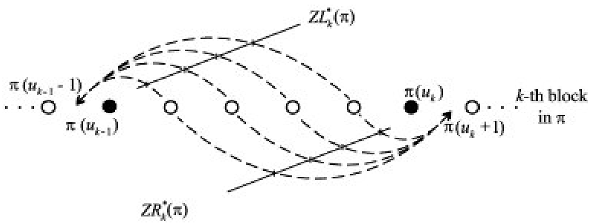
Perturbation



- perform all *inserts* among all the blocks that have $\Delta < 0$
- activated after `MaxIdleIter` idle iterations

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Tabu Search: the final algorithm:

Initialization	:	$\pi = \pi_0$, $C^* = C_{max}(\pi)$, set iteration counter to zero.
Searching	:	Create UR_k and UL_k (set of non tabu moves)
Selection	:	Find the best move according to lower bound Δ. Apply move. Compute true $C_{max}(\delta(\pi))$. If improving compare with C^* and in case update. Else increase number of idle iterations.
Perturbation	:	Apply perturbation if `MaxIdleIter` done.
Stop criterion	:	Exit if `MaxIter` iterations are done.