DMP204
SCHEDULING,
TIMETABLING AND ROUTING

Lecture 25

Vehicle Routing Mathematical Programming

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Outline

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Basic Models

- arc flow formulation
  integer variables on the edges counting the number of time it is traversed
  one, two or three index variables
- set partitioning formulation
- multi-commodity network flow formulation for VRPTW
  integer variables representing the flow of commodities along the paths traveled by the vehicles and
  integer variables representing times

Two index arc flow formulation

$$\min \quad \sum_{i\in V}\sum_{j\in V} c_{ij}x_{ij} \quad (1)$$

$$\text{s.t.} \quad \sum_{i\in V} x_{ij} = 1 \quad \forall j \in V \setminus \{0\} \quad (2)$$

$$\sum_{j\in V} x_{ij} = 1 \quad \forall i \in V \setminus \{0\} \quad (3)$$

$$\sum_{i\in V} x_{i0} = K \quad (4)$$

$$\sum_{j\in V} x_{0j} = K \quad (5)$$

$$\sum_{i\in S}\sum_{i\notin S} x_{ij} \geq r(S) \quad \forall S \subseteq V \setminus \{0\}, S \neq \emptyset \quad (6)$$

$$x_{ij} \in \{0,1\} \quad \forall i,j \in V \quad (7)$$

One index arc flow formulation

$$\min \quad \sum_{e\in E} c_e x_e \quad (8)$$

$$\text{s.t.} \quad \sum_{e\in\delta(i)} x_e = 2 \quad \forall i \in V \setminus \{0\} \quad (9)$$

$$\sum_{e\in\delta(0)} x_e = 2K \quad (10)$$

$$\sum_{e\in\delta S} x_e \geq 2r(S) \quad \forall S \subseteq V \setminus \{0\}, S \neq \emptyset \quad (11)$$

$$x_e \in \{0,1\} \quad \forall e \notin \delta(0) \quad (12)$$

$$x_e \in \{0,1,2\} \quad \forall e \in \delta(0) \quad (13)$$

Three index arc flow formulation

$$\min \quad \sum_{i\in V}\sum_{j\in V} c_{ij} \sum_{k=1}^{K} x_{ijk} \quad (14)$$

$$\text{s.t.} \quad \sum_{k=1}^{K} y_{ik} = 1 \quad \forall i \in V \setminus \{0\} \quad (15)$$

$$\sum_{k=1}^{K} y_{0k} = 1 \quad (16)$$

$$\sum_{j\in V} x_{ijk} = \sum_{j\in V} x_{jik} = y_{ik} \quad \forall i \in V, k = 1,\ldots,K \quad (17)$$

$$\sum_{i\in V} d_i y_{ik} \leq C \quad \forall k = 1,\ldots,K \quad (18)$$

$$\sum_{i\in S}\sum_{i\notin S} x_{ijk} \geq y_{hk} \quad \forall S \subseteq V \setminus \{0\}, h \in S, k = 1,1,\ldots,K \quad (19)$$

$$y_{ik} \in \{0,1\} \quad \forall i \in V, k = 1,\ldots,K \quad (20)$$

$$x_{ijk} \in \{0,1\} \quad \forall i,j \in V, k = 1,\ldots,K \quad (21)$$

What can we do with these integer programs?

- plug them into a commercial solver and try to solve them
- preprocess them
- determine lower bounds
  - solve the linear relaxation
  - combinatorial relaxations
    relax some constraints and get an easy solvable problem
  - Lagrangian relaxation
  - polyhedral study to tighten the formulations
- upper bounds via heuristics
- branch and bound
- cutting plane
- branch and cut
- column generation (via reformulation)
- branch and price
- Dantzig Wolfe decomposition
- upper bounds via heuristics

Combinatorial Relaxations

Lower bounding via combinatorial relaxations

Relax: capacity cut constraints (CCC)
or generalized subtour elimination constraints (GSEC) Consider both

ACVRP and SCVRP

- Relax CCC in 2-index formulation
  obtain a transportation problem
  Solution may contain isolated circuits and exceed vertex capacity

- Relax CCC in 1-index formulation
  obtain a b-matching problem

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = b_i && \forall i \in V\{0\} \\
& x_e \in \{0,1\} && \forall e \notin \delta(0) \\
& x_e \in \{0,1,2\} && \forall e \in \delta(0)
\end{aligned}
$$

  Solution has same problems as above

- relax in two index flow formulation:

$$
\begin{aligned}
\min \quad & \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \in V} x_{ij} = 1 && \forall j \in V \setminus \{0\} \\
& \sum_{j \in V} x_{ij} = 1 && \forall i \in V \setminus \{0\} \\
& \sum_{i \in V} x_{i0} = K \\
& \sum_{j \in V} x_{0j} = K \\
& \sum_{i \in S} \sum_{j \notin S} x_{ij} \geq r(S)1 && \forall S \subseteq V \setminus \{0\}, S \neq \emptyset \\
& x_{ij} \in \{0,1\} && \forall i,j \in V
\end{aligned}
$$

  K-shortest spanning arborescence problem

- relax in two index formulation

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 2 && \forall i \in V \setminus \{0\} \\
& \sum_{e \in \delta(0)} x_e = 2K \\
& \sum_{e \in \delta S} x_e \geq 2r(S) && \forall S \subseteq V \setminus \{0\}, S \neq \emptyset \\
& x_e \in \{0,1\} && \forall e \notin \delta(0)
\end{aligned}
$$

  K-tree: min cost set of $n + K$ edges spanning the graph with degree 2K at the depot.

- Lagrangian relaxation of the sub tour constraints iteratively added after violation recognized by separation procedure.
  Subgradient optimization for the multipliers.

Branch and Cut

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e && && (22) \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = 2 && \forall i \in V \setminus \{0\} && (23) \\
& \sum_{e \in \delta(0)} x_e = 2K && && (24) \\
& \sum_{e \in \delta S} x_e \geq 2\left\lceil \frac{d(S)}{C} \right\rceil && \forall S \subseteq V \setminus \{0\}, S \neq \emptyset && (25) \\
& x_e \in \{0,1\} && \forall e \notin \delta(0) && (26) \\
& x_e \in \{0,1,2\} && \forall e \in \delta(0) && (27)
\end{aligned}
$$

Branch and Cut

- Let $LP(\infty)$ be linear relaxation of IP
- $z_{LP(\infty)} \leq z_{IP}$
- Problems if many constraints
- Solve $LP(h)$ instead and add constraints later
- If $LP(h)$ has integer solution then we are done, that is optimal
  If not, then $z_{LP(h)} \leq z_{LP(h+1)} \leq z_{LP(\infty)} \leq z_{IP}$
- Crucial step: separation algorithm given a solution to $LP(h)$ it tell us if some constraints are violated.

If the procedure does not return an integer solution we proceed by branch and bound

Problems with B&C:

i) no good algorithm for the separation phase
it may be not exact in which case bounds relations still hold and we can go on with branching

ii) number of iterations for cutting phase is too high

iii) program unsolvable because of size

iv) the tree explodes

The main problem is (iv).
Worth trying to strengthen the linear relaxation by inequalities that although unnecessary in the integer formulation force the optimal solution of LP and IP to get closer. ➨ Polyhedral studies

Set Covering Formulation

$\mathcal{R} = \{1, 2, \ldots, R\}$ index set of routes

$$a_{ir} = \begin{cases} 1 & \text{if costumer } i \text{ is served by } r \\ 0 & \text{otherwise} \end{cases}$$

$$x_r = \begin{cases} 1 & \text{if route } r \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$\min \sum_{r \in \mathcal{R}} c_r x_r \qquad (28)$$

$$\text{s.t.} \sum_{r \in \mathcal{R}} a_{ir} x_r \geq 1 \qquad \forall i \in V \quad (29)$$

$$\sum_{r \in \mathcal{R}} x_r \leq K \qquad (30)$$

$$x_r \in \{0, 1\} \qquad \forall r \in \mathcal{R} \quad (31)$$

$$(32)$$

Solving the SCP integer program

Branch and bound

- Generate routes such that:
  - they are good in terms of cost
  - they reduce the potential for fractional solutions
- constraint branching [Ryan, Foster, 1981]

$$\exists \text{ constraints } r_1, r_2 : 0 < \sum_{j \in J(r_1, r_2)} x_j < 1$$

$J(r_1, r_2)$ all columns covering r_1, r_2 simultaneously. Branch on:

$$\sum_{j \in J(r_1, r_2)} x_j \leq 0 \qquad / \qquad \backslash \qquad \sum_{j \in J(r_1, r_2)} x_j \geq 1$$

Solving the SCP linear relaxation

Column Generation Algorithm

Step 1 Generate an initial set of columns $\mathcal{R}'$

Step 2 Solve problem P' and get optimal primal variables, $\bar{x}$, and optimal dual variables, $(\bar{\pi}, \bar{\theta})$

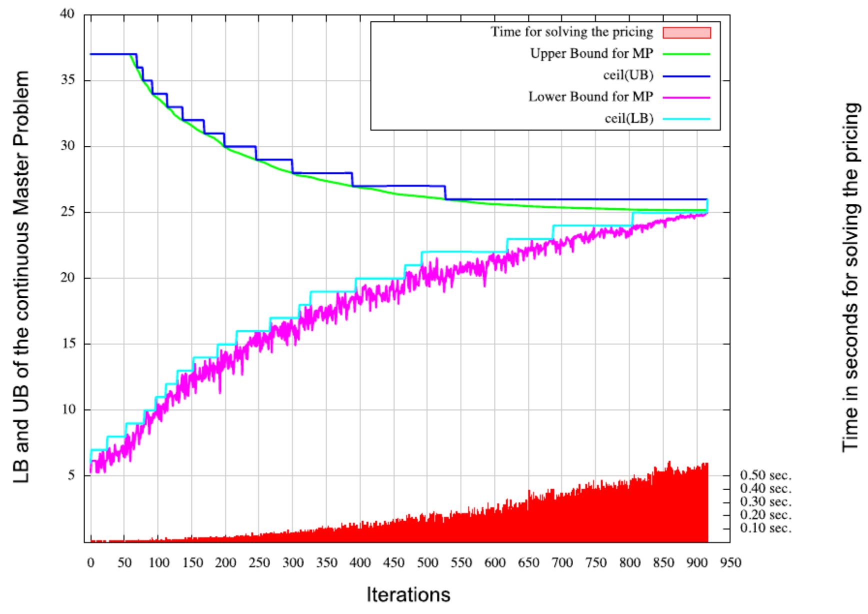
Step 3 Solve problem CG, or identify routes $r \in \mathcal{R}$ satisfying $\bar{c}_r < 0$

Step 4 For every $r \in \mathcal{R}$ with $\bar{c}_r < 0$ add the column r to $\mathcal{R}'$ and go to Step 2

Step 5 If no routes r have $\bar{c}_r < 0$, i.e., $\bar{c}_{min} \geq 0$ then stop.

In most of the cases we are left with a fractional solution

Convergence in CG



[plot by Stefano Gualandi, Milan Un.]

Solving the SCP integer program:

- cutting plane
- branch and price

Cutting Plane Algorithm

Step 1 Generate an initial set $\mathcal{R}'$ of columns

Step 2 Solve, using column generation, the problem P' (linear programming relaxation of P)

Step 3 If the optimal solution to P' is integer stop.
Else, generate cutting plane separating this fractional solution.
Add these cutting planes to the linear program P'

Step 4 Solve the linear program p'. Goto Step 3.

Is the solution to this procedure overall optimal?

Cuts

Intersection graph $G = (V, E)$ where V are the routes and E is made by the links between routes that intercept
Independence set in G is a collection of routes where each customer is visited only once.

Clique constraints

$$\sum_{r \in K} \bar{x}_r \leq 1 \qquad \forall \text{ cliques } K \text{ of } G$$

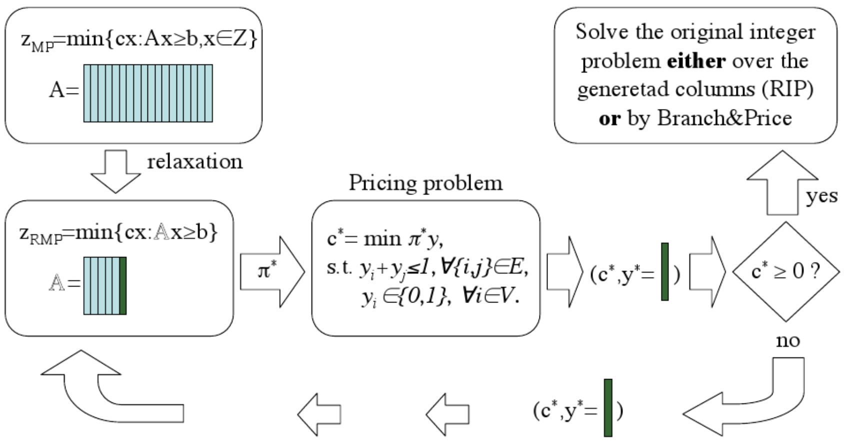
Cliques searched heuristically

Odd holes

Odd hole: odd cycle with no chord

$$\sum_{r \in H} \bar{x}_r \leq \frac{|H| - 1}{2} \qquad \forall \text{ odd holes } H$$

Generation via layered graphs



[illustration by Stefano Gualandi, Milan Un.]
(the pricing problem is for a GCP)

Branch and price

- it must be possible to incorporate information on the node in the column generation procedure
- easy to incorporate $x_r = 1$, just omit nodes in S_r from generation; but not clear how to impose $x_r = 0$.
- different branching: on the edges: $x_{ij} = 1$ then in column generation set $c_{ij} = -\infty$; if $x_{ij} = 0$ then $c_{ij} = \infty$

Implementation details

- throw out from LP columns that have not been basic for a long time
- good procedures to generate good columns
- generate columns that are disjoint, collectively exhaustive and of minimal cost