DMP204
SCHEDULING,
TIMETABLING AND ROUTING

Lecture 26

Vehicle Routing
Mathematical Programming

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Outline

Outline

VRPTW

$$
\begin{aligned}
\min \quad & \sum_{k \in K} \sum_{(i,j) \in A} c_{ij} x_{ijk} && && (1)\\
\text{s.t.} \quad & \sum_{k \in K} \sum_{(i,j) \in \delta^+(i)} x_{ijk} = 1 && \forall i \in V && (2)\\
& \sum_{(i,j) \in \delta^+(0)} x_{ijk} = \sum_{(i,j) \in \delta^-(0)} x_{ijk} = 1 && \forall k \in K && (3)\\
& \sum_{(i,j) \in \delta^-(i)} x_{jik} - \sum_{(i,j) \in \delta^+(i)} x_{ijk} = 0 && i \in V, k \in K && (4)\\
& \sum_{(i,j) \in A} d_i x_{ijk} \leq C && \forall k \in K && (5)\\
& x_{ijk}(w_{ik} + t_{ij}) \leq w_{jk} && \forall k \in K, (i,j) \in A && (6)\\
& a_i \leq w_{ik} \leq b_i && \forall k \in K, i \in V && (7)\\
& x_{ijk} \in \{0, 1\} && && (8)
\end{aligned}
$$

Pre-processing

- Arc elimination
  - $a_i + t_{ij} > b_j$ ➜ arc (i, j) cannot exist
  - $d_i + d_j > C$ ➜ arcs (i, j) and (j, i) cannot exist

- Time windows reduction
  - $[a_i, b_i] \leftarrow [\max\{a_0 + t_{0i}, a_i\}, \min\{b_{n+1} - t_{i,n+1}, b_i\}]$ why?

- Time windows reduction:
  - Iterate over the following rules until no one applies anymore:

1) Minimal arrival time from predecessors:

$$a_l = \max\left\{a_l, \min\left\{b_l, \min_{(i,l)}\{a_i + t_{il}\}\right\}\right\}.$$

2) Minimal arrival time to successors:

$$a_l = \max\left\{a_l, \min\left\{b_l, \min_{(l,j)}\{a_j - t_{lj}\}\right\}\right\}.$$

3) Maximal departure time from predecessors:

$$b_l = \min\left\{b_l, \max\left\{a_l, \max_{(i,l)}\{b_i + t_{il}\}\right\}\right\}.$$

4) Maximal departure time to successors:

$$b_l = \min\left\{b_l, \max\left\{a_l, \max_{(l,j)}\{b_j - t_{lj}\}\right\}\right\}.$$

Lower Bounds

- Combinatorial relaxation
  reduce to network flow problem

- Lagrangian relaxation
  not very good because easy to not satisfy the capacity and time windows constraints

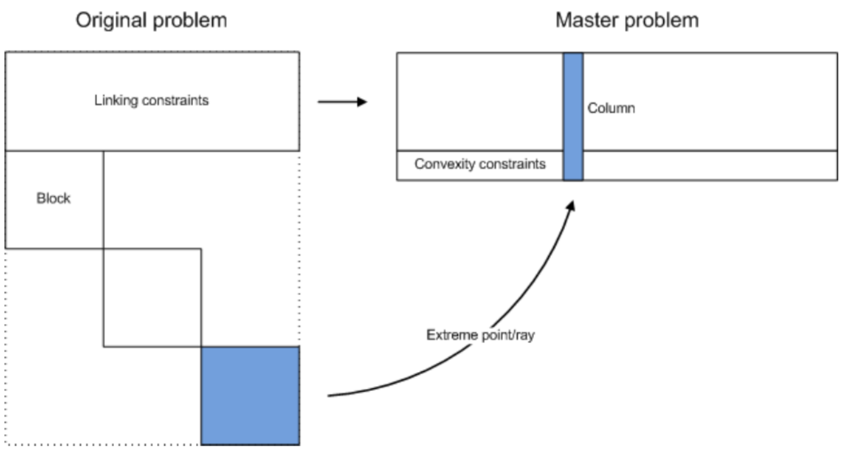
Dantzig Wolfe Decomposition

The VRPTW has the structure:

$$\begin{aligned}
\min \quad & c^k x^k \\
& \sum_{k \in K} A^k x^k \leq b \\
& D^k x^k \leq d^k && \forall k \in K \\
& x^k \in \mathbb{Z} && \forall k \in K
\end{aligned}$$

Dantzig Wolfe Decomposition

Illustrated with matrix blocks



[illustration by Simon Spoorendonk, DIKU]

Linking constraint in VRPTW is $\sum_{k \in K} \sum_{(i,j) \in \delta^+(i)} x_{ijk} = 1, \quad \forall i$. The description of the block $D^k x^k \le d^k$ is all the rest:

$$\sum_{(i,j) \in A} d_i x_{ij} \le C \tag{9}$$

$$\sum_{j \in V} x_{0j} = \sum_{i \in V} x_{i,n+1} = 1 \tag{10}$$

$$\sum_{i \in V} x_{ih} - \sum_{j \in V} x_{hj} = 0 \qquad \forall h \in V \tag{11}$$

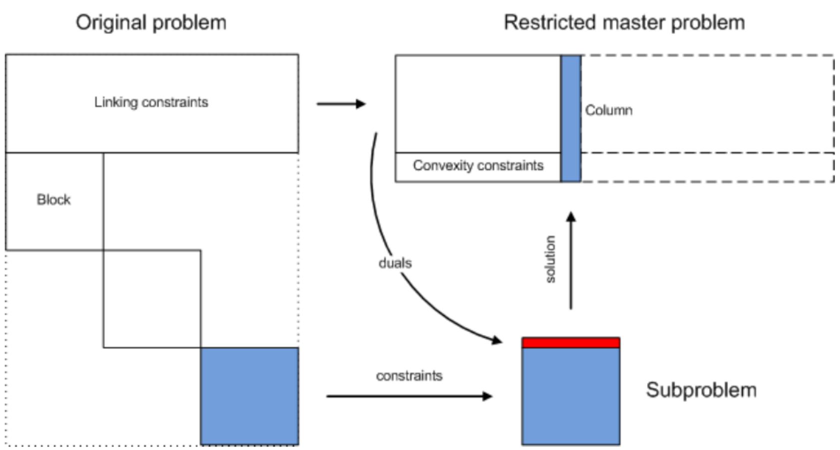
$$w_i + t_{ij} - M_{ij}(1 - x_{ij}) \le w_j \qquad \forall (i,j) \in A \tag{12}$$

$$a_i \le w_i \le b_i \qquad \forall i \in V \tag{13}$$

$$x_{ij} \in \{0,1\} \tag{14}$$

where we omitted the index k because, by the assumption of homogeneous fleet, all blocks are equal.

Dantzig Wolfe Decomposition



[illustration by Simon Spoorendonk, DIKU]

Master problem

A Set Partitioning Problem

$$\min \sum_{p \in \mathcal{P}} c_{ij} \alpha_{ijp} \lambda_p \tag{15}$$

$$\sum_{p \in \mathcal{P}} \sum_{(i,j) \in \delta^+(i)} \alpha_{ijp} \lambda_p = 1 \qquad \forall i \in V \tag{16}$$

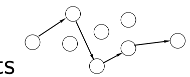
$$\lambda_p = \{0,1\} \qquad \forall p \in \mathcal{P} \tag{17}$$

where $\mathcal{P}$ is the set of valid paths and $\alpha_{ijp} = \begin{cases} 0 & \text{if } (i,j) \not\in p \\ 1 & \text{otherwise} \end{cases}$

Subproblem

Elementary Shortest Path Problem with Resource Constraints (ESPPRC)

- arcs modified with duals (possible negative costs), NP-hard
- find shortest path without violating resource limits



Subproblem

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} \hat{c}_{ij} x_{ij} && && (18)\\
\text{s.t.} \quad & \sum_{(i,j)\in A} d_i x_{ij} \le C && && (19)\\
& \sum_{j\in V} x_{0j} = \sum_{i\in V} x_{i,n+1} = 1 && && (20)\\
& \sum_{i\in V} x_{ih} - \sum_{j\in V} x_{hj} = 0 && \forall h \in V && (21)\\
& w_i + t_{ij} - M_{ij}(1 - x_{ij}) \le w_j && \forall (i,j) \in A && (22)\\
& a_i \le w_i \le b_i && \forall i \in V && (23)\\
& x_{ij} \in \{0,1\} && && (24)
\end{aligned}
$$

Subproblem

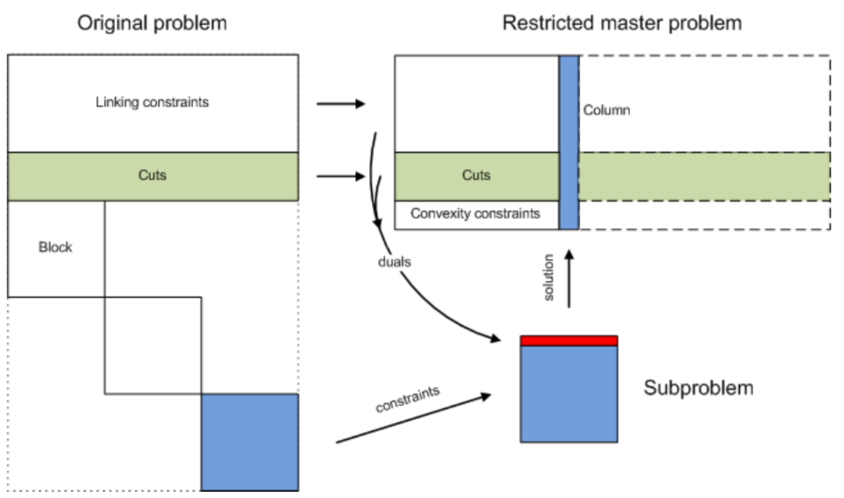
Solution Approach:

- Solved by dynamic programming. Algorithms maintain labels at vertices and remove dominated labels. Domination rules are crucial.
- relaxing and allowing cycles the problem can be sovled in pseudo-polynomial time.
  Negative cycles are however limited by the resource constraints
- optimal solution has only elementary routes if triangle inequality holds.
  Otherwise post-process by cycle elimination procedures

For details see chp. 2 of [B11]

Branch and Bound

Cuts in the original three index problem formulation (before DWD)



[illustration by Simon Spoorendonk, DIKU]

Branching

- branch on $\sum_k x_{ijk}$
  choose a candidate not close to 0 or 1
  $\max c_{ij} \min\{x_{ijk}, 1 - x_{ijk}\}$
- branch on time windows
  split time windows s.t. at least one route becomes infeasible
  compute $[l_i^r, u_i^r]$ (earliest latest) for the current fractional flow
  $L^i = \max_{\text{fract. routes } r} \{l_i^r\} \quad \forall i \in V$
  $U^i = \max_{\text{fract. routes } r} \{u_i^r\} \quad \forall i \in V$
  if $L_i > U_i$ ➨ at least two routes have disjoint feasibility intervals