EX 7. Ass. 1

a) 4 variables, hence $2^4 = 16$ models

b) An implication is false if premise is true and conclusion false. There are 4 models where $R \wedge C$ is ~~false~~ true. The negated conclusion is $\neg(\neg O \wedge \neg B)$ which is just $O \vee B$, and this is true in 3 of the 4 models.

c) Yes. It is equivalent to

$R \wedge C \Rightarrow \neg O$ and $R \wedge C \Rightarrow \neg B$

In clause form, these become

$\neg R \vee \neg C \vee \neg O$ and $\neg R \vee \neg C \vee \neg B$,

These clause have zero positive literals, and hence are Horn.

d) To prove that A does not entail B, one simply has to provide a model where A is true and B is false. The model is $R, C, \neg B, O$

EX 9 Ass. 1

a) Prove $A \Rightarrow B$ and $B \Rightarrow A$.
In both cases put in CNF and use resolution by unification.

b) $A: \forall x\ [\exists y\ P(x,y)] \Rightarrow Q(x)$

$\forall x\ \neg[\exists y\ P(x,y)] \vee Q(x)$

$\forall x\ [\forall y\ \neg P(x,y)] \vee Q(x)$

$\forall x, y\ \neg P(x,y) \vee Q(x)$

$\neg P(x,y) \vee Q(x)$

$B: \forall x, y\ \ P(x,y) \Rightarrow Q(x)$

$\forall x, y\ \ \neg P(x,y) \vee Q(x)$

$\neg P(x,y) \vee Q(x)$

$\neg B: \neg[\forall x, y\ P(x,y) \Rightarrow Q(x)]$

$\neg[\forall x, y\ \neg P(x,y) \vee Q(x)]$

$\exists x, y\ \neg[\neg P(x,y) \vee Q(x)]$

$\exists x, y\ [P(x,y) \wedge \neg Q(x)]$

$P(a,H) \wedge \neg Q(a)$

$A \models B \leadsto A \wedge \neg B$

conjunction of 3 clauses → resolution

$$\frac{\neg P(x,y) \vee Q(x) \qquad P(a,H)}{\therefore Q(a) \quad \theta = \{x/a, y/H\}}
\qquad
\frac{\neg P(x,y) \vee Q(x) \qquad \neg Q(a)}{\therefore \neg P(a,y) \quad \theta = \{x/a\}}
\qquad
\frac{\neg Q(a) \qquad Q(a)}{\therefore \emptyset}$$

$A \wedge \neg B$ is FALSE hence $A \models B$

Ex 1 Ass. 2

a) $P(\text{toothache}) = 0{,}108 + 0{,}012 + 0{,}016 + 0{,}064 = 0{,}120 + 0{,}08 = 0{,}2$

b) $P(\text{Cavity}) = \langle 0{,}108 + 0{,}012 + 0{,}072 + 0{,}008 ; 0{,}016 + 0{,}064 + 0{,}144 + 0{,}576 \rangle =$

$= \langle 0{,}2 ; 0{,}8 \rangle$

c) $P(\text{Toothache} \mid \text{cavity}) = \left\langle \frac{0{,}108 + 0{,}012}{0{,}2} ; \frac{0{,}072 + 0{,}008}{0{,}2} \right\rangle =$

$= \left\langle \frac{0{,}120}{0{,}2} ; \frac{0{,}080}{0{,}2} \right\rangle =$

$\frac{P(\text{Toot}, \text{cavity})}{P(\text{cavity})}$ $= \langle 0{,}6 ; 0{,}4 \rangle$

d) $P(\text{Cavity} \mid \text{toothache} \vee \text{catch}) = \frac{P(\text{Cavity}, \text{toothache} \vee \text{catch})}{P(\text{toothache} \vee \text{catch})} =$

$= \langle 0{,}4615 ; 0{,}5384 \rangle$

Ex. 2 Ass 2

$P(A,B \mid C) = P(A \mid C)\, P(B \mid C)$ cond. independence hence:

$P(A \mid B, C) = P(A \mid C)$ and $P(B \mid A, C) = P(B \mid C)$

Att. this has no sense: $P(A,B \mid C) = P(A \mid C, B \mid C) = P(A \mid C \mid B \mid C) \cdot P(B \mid C)$

However, also:

$\frac{P(A,B,C)}{P(B,C)} = P(A \mid B, C)$ *

product rule

$\frac{P(A,B,C)}{P(B,C)} \underset{\text{Cond. prob.}}{=} \frac{P(A,B \mid C)\, P(C)}{P(BC)} \underset{\text{by hypothesis}}{=} \frac{P(A \mid C)\, P(B \mid C)\, P(C)}{P(BC)} \underset{\text{prod. rule inverse}}{=} \frac{P(A \mid C)\, \cancel{P(BC)}}{\cancel{P(BC)}}$

Ex. 3 Ass. 2

a) $P(x, y \mid e) = P(x \mid y, e)\, P(y \mid e)$

$P(x, y \mid e) = \frac{P(x, y, e)}{P(e)} = \frac{P(x \mid y, e)\, P(y, e)}{P(e)} = P(x \mid y, e)\, P(y \mid e)$

b) $P(y \mid x, e) = \frac{P(x \mid y, e)\, P(y \mid e)}{P(x \mid e)}$

$P(y \mid x, e) = \frac{P(y, x, e)}{P(x, e)} = \frac{P(x \mid y, e)\, P(y, e)}{P(x, e)} =$

Cond. prob.

$= \frac{P(x \mid y, e) \cdot P(y \mid e)\, P(e)}{P(x \mid e) \cdot P(e)} \underset{\text{Cond. prob}}{=} \frac{P(x \mid y, e)\, P(y \mid e)}{P(x \mid e)}$

EX. 4 Ass 2

$p(s|m) = 0{,}5$

$p(m) = 1/50000$

$p(s) = 1/20$

$$p(m|s) = \frac{p(s|m) \cdot p(m)}{p(s)}$$

$$p(M|s) = \alpha < p(s|m) \cdot p(m), p(s|\neg m) \cdot p(\neg m) > =$$

$$= \alpha < 0{,}5 \cdot \frac{1}{50000} ; 0{,}05 \left(1 - \frac{1}{50000}\right) > =$$

$$= \alpha < 1 \cdot 10^{-5} ; 0{,}05 \cdot 0{,}99998 > =$$

$$= < 0{,}0002 ; 0{,}9998 >$$

EX. 5 Ass. 2

E_A A is executed

F_B B is freed, the guard brings him the message

$$P(E_A|F_B) \overset{\text{Bayes}}{=} \frac{\overbrace{P(F_B|E_A)}^{\to \frac{1}{2}} \cdot \overbrace{P(E_A)}^{\to \frac{1}{3}}}{\underbrace{P(F_B)}_{\to ?}} = \frac{1}{3}$$

$$P(F_i|E_k) = \begin{cases} 0 & i = k \\ \frac{1}{2} & i \neq k \\ 1 & i = B \quad k = c \end{cases}$$

$$P(F_B) = P(F_B, E_A) + P(F_B, E_B) + P(F_B, E_c) =$$

$$= P(F_B|E_A) P(E_A) + P(F_B|E_B) P(E_B) + P(F_B|E_c) P(E_c)$$

$$= \frac{1}{2} \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{3} = \frac{1}{2}$$

$$1 = \underbrace{P(E_A|F_B)}_{\frac{1}{3}} + \underbrace{P(E_B|F_B)}_{\frac{P(F_B|E_B) \cdot P(E_B)}{P(F_B)} = 0} + P(E_c|F_B)$$

$$P(E_c|F_B) = \frac{2}{3}$$

EX. 6 ASS. 2

$p(d_1) = 0{,}2$ $\qquad p(a_1|d_1) = 0{,}2$

$p(d_2) = 0{,}4$ $\qquad p(a_1|d_2) = 0{,}4$

$p(d_3) = 0{,}4$ $\qquad p(a_1|d_3) = 0{,}3$

$$p(d_2|a_1) = \frac{p(a_1|d_2)\,p(d_2)}{p(a_1)} = \frac{0{,}16}{0{,}32} = 0{,}5$$

$$p(a_1) = \underbrace{\sum_i p(a_1 d_i)}_{\text{marginal}} = \underbrace{\sum p(a_1|d_i)\,p(d_i)}_{\text{cond. prob.}}$$

EX 7. ASS 2

a) i) incorrect: n° of stars not indep. of the focus given the measurements

ii) correct

iii) incorrect: $M_1 \to M_2$? However, could still be plausible

b) ii) it has less parameters than iii) and is correct

c) $p(M_1|N)$ $\qquad N \in \{1,2,3\}$ $\quad M_1 \in \{0,1,2,3,4\}$

marg. + cond. prob.

$$p(M_1|N) = p(M_1|N, \neg F_1)\,p(\neg F_1) + p(M_1|N, F_1)\,p(F_1)$$

f prob. of counting −3 because out of focus

e prob. of counting ± 1 if in focus

1-2e count is accurate

	N = 1	N = 2	N = 3	N = 4	N = 5
$M_1 = 0$	$e(1-f)+f$	f	f		
$M_1 = 1$	$(1-2e)(1-f)$	$e(1-f)$	0	;	;
$M_1 = 2$	$e(1-f)$	$(1-2e)(1-f)$	$e(1-f)$		
$M_1 = 3$	0	$e(1-f)$	$(1-2e)(1-f)$		
$M_1 = 4$	0	0	$e(1-f)$		
	$\Sigma = 1$	$\Sigma = 1$	$\Sigma = 1$		

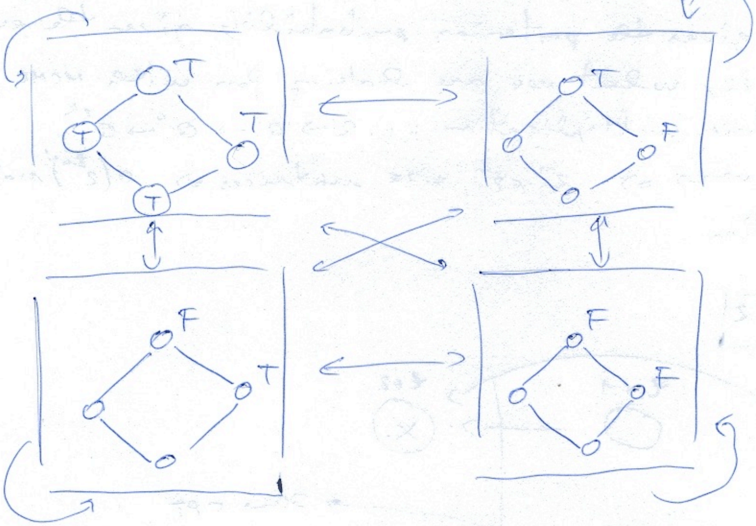
d) Try all $N = 1, 2 \ldots n$ and see for which $M_1 = 1$ and $M_2 = 3$ are consistent. Or, try all possible focus states and deduce values for N. Answer: $N = 2, 4$ and ≥ 6

e) we need to know $p(N)$ $\qquad p(N|M_1=1, M_2=3) = \dfrac{p(M_1=1, M_2=3|N)\,p(N)}{p(M_1=1)\,p(M_2=3)}$

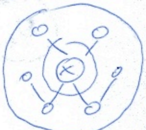
$\boxed{\text{Ex. 8 Ass. 2}}$

$P(Rain \mid Sprinkler = T, WG = T)$

a) 4:



b) Sampling distr. conditional to Markov blanket



$$P(C \mid r, s) = \alpha P(C) P(s \mid C) P(r \mid C) = \langle 0{,}5 ; 0{,}5 \rangle \langle 0{,}1 ; 0{,}5 \rangle \langle 0{,}8 ; 0{,}2 \rangle = \langle 4/9 ; 5/9 \rangle$$

$$P(C \mid \neg r, s) = \alpha P(C) P(s \mid C) P(\neg r \mid C) = \quad = \langle 1/21 ; 20/21 \rangle$$

$$P(R \mid c, s, w) = \alpha P(R \mid c) P(w \mid s\, R) = \alpha \langle \quad \rangle \langle \quad \rangle = \langle 22/27 , 5/27 \rangle$$

$$P(R \mid \neg c, s, w) = \alpha P(R \mid \neg c) P(w \mid s\, R) = \alpha \langle \quad \rangle \langle \quad \rangle = \langle 11/51 ; 40/51 \rangle$$

Transition ~~matrix~~ equal. prob. of sampling either of the two var.

loop $\quad q((c,r) \to (c,r)) = 0{,}5 P(c \mid r, s) + 0{,}5 P(r \mid c, s, w) = 17/27$

one var. changes $\quad q((c,r) \to (c, \neg r)) = 0{,}5 P(\neg r \mid c, s, w) = 5/54$

both var. changes $\quad q((c,r) \to (\neg c, \neg r)) = 0$ cannot occur

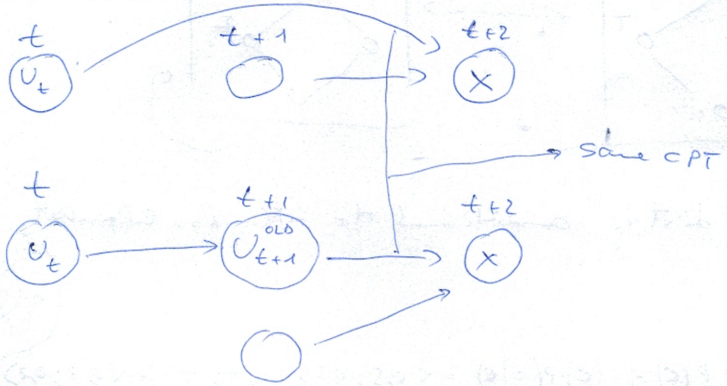
$$\begin{array}{c|cccc} & (c,r) & (c,\neg r) & (\neg c, r) & (\neg c, \neg r) \\ \hline (c,r) & 17/27 & 5/54 & 5/18 & 0 \\ (c,\neg r) & 11/27 & 22/189 & 0 & \vdots \\ (\neg c, r) & \vdots & 0 & \vdots & \vdots \\ (\neg c, \neg r) & 0 & \vdots & \vdots & \vdots \end{array}$$

c) Q^2 prob. of going from each state to any other in 2 steps.

d) Q^n $n \to \infty$ long term probability, for ergodic process this gives the posterior probability given the evidence, ie, what we are looking for with MCMC

e) can do matrix multiplication : $Q \to Q^2 \to Q^4 \to Q^{2^k}$
but if n vars $\Rightarrow$ $2^n \times 2^n$ size matrices $\Rightarrow$ $O(2^{3n})$ matrix multiplication

$\boxed{\text{EX. 15.1=9 ASS. 2}}$



introduce a new var in t+1, U^{OLD}_{t+1} with CPT $\begin{array}{|cc} 1 & 0 \\ 0 & 1 \end{array}$
with respect to U_t, ie, $p(U^{OLD}_{t+1} | U_t)$

no effective increase in parameters.

$\boxed{\text{Ex 10} \quad \text{Ass 2}}$

A = activity

$P(A = \text{walk}) = 0{,}5$

W = weather

$$P(x|y,e) = P(x,y,e) \cdot P(y,e) = \frac{P(x,y|e)P(e)}{P(y,e)} = \frac{P(x,y|e)}{P(y|e)}$$

$$P(W_2 = \text{Rainy} \mid W_1 = \text{Sunny}, A_2 = \text{Read}) =$$

$$= \frac{P(W_2 = R, W_1 = S \mid A_2 = R)}{P(W_1 = S \mid A_1 = R)} = \quad W_1 \,\&\, A_1 \text{ are indep.}$$

$$= \frac{P(W_2 = R, W_1 = S \mid A_2 = R)}{P(W_1 = S)} = \quad \text{Bayes}$$

$$= \frac{P(A_2 = R \mid W_1 = S, W_2 = R)\, P(W_2 = R, W_1 = S)}{P(W_1 = S)\, P(A_2 = R)} = \quad \text{Markov}$$

$$= \frac{P(A_2 = R \mid W_2 = R)\, P(W_2 = R, W_1 = S)}{P(W_1 = S)\, P(A_2 = R)} = \quad P(A,B) = P(A|B)P(B)$$

$$= \frac{\_\_\_\_ \; P(W_2 = R \mid W_1 = S)\, \cancel{P(W_1 = S)}}{\cancel{P(W_1 = S)}\, P(A_2 = R)} =$$

$$= \frac{0{,}8 \cdot 0{,}05}{0{,}5} = 0{,}08$$

b) $P(W_3 = F \mid W_1 = F, A_2 = R, A_3 = W) =$

$= P(W_2 = F, W_3 = F \mid W_1 = F, A_2 = R, A_3 = W) +$

$+ P(W_2 = S, W_3 = F \mid W_1 = F, A_2 = R, A_3 = W) +$

$+ P(W_2 = R, W_3 = F \mid \quad \_\_\_ \quad) =$ by similar passages as above

$$= \frac{P(A_3 = W \mid W_3 = F)\, P(A_2 = R \mid W_2 = F)\, P(W_3 = F \mid W_2 = F)\, P(W_2 = F \mid W_1 = F)\, \cancel{P(W_1 = F)}}{P(A_3 = W)\, P(A_2 = R)\, \cancel{P(W_1 = F)}} +$$

+ similar passages for the other 2 terms.

$$= \frac{0{,}7 \cdot 0{,}8 \cdot 0{,}5 \cdot 0{,}15}{0{,}5 \cdot 0{,}5 \cdot 1} + \_\_ + \dots$$

$= 0{,}119$