**DM204**, 2010
SCHEDULING, TIMETABLING AND ROUTING

Lecture 26
## Workforce Timetabling

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# Course Overview



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  - ✔ Scheduling classification
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# Outline







# Outline

**Shift:** consecutive working hours

**Roster:** shift and rest day patterns over a fixed period of time (a week or a month)

Two main approaches:

- coordinate the design of the rosters and the assignment of the shifts to the employees, and solve it as a single problem.
- consider the scheduling of the actual employees only after the rosters are designed, solve two problems in series.

Features to consider: rest periods, days off, preferences, availabilities, skills.





Workforce Scheduling:

1. Crew Scheduling and Rostering
2. Employee Timetabling

1. Crew Scheduling and Rostering is workforce scheduling applied in the transportation and logistics sector for enterprises such as airlines, railways, mass transit companies and bus companies (pilots, attendants, ground staff, guards, drivers, etc.)

   The peculiarity is finding logistically feasible assignments.





2. Employee timetabling (aka labor scheduling) is the operation of assigning employees to tasks in a set of shifts during a fixed period of time, typically a week.

Examples of employee timetabling problems include:

- assignment of nurses to shifts in hospitals
- assignment of workers to cash registers in a large store
- assignment of phone operators to shifts and stations in a service-oriented call-center

Differences with Crew scheduling:

- no need to travel to perform tasks in locations
- start and finish time not predetermined



# Outline

## Shift Scheduling



Creating daily shifts:

- during each period, $b_i$ persons required
- decide working rosters made of $m$ time intervals not necessarily identical
- $n$ different shift patterns (columns of matrix $A$) each with a cost $c$

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax \geq b \\ & x \geq 0 \text{ and integer} \end{aligned}$$

$$\min c^T x$$

$$\begin{matrix} 10am - 11pm \\ 11am - 12am \\ 12am - 1pm \\ 1pm - 2pm \\ 2pm - 3pm \\ 3pm - 4pm \\ 4pm - 5pm \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} x \geq \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

$$x \geq 0 \text{ and integer}$$



## $(k, m)$-cyclic Staffing Problem



Assign persons to an $m$-period cyclic schedule so that:

- requirements $b_i$ are met
- each person works a shift of $k$ consecutive periods and is free for the other $m - k$ periods. (periods $1$ and $m$ are consecutive)

and the cost of the assignment is minimized.

$$\begin{aligned} \min \quad & c^T x \\ st \quad & \begin{matrix} Monday \\ Tuesday \\ Wednesday \\ Thursday \\ Friday \\ Saturday \\ Sunday \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} x \geq \begin{bmatrix} 3 \\ 4 \\ 6 \\ 4 \\ 7 \\ 8 \\ 7 \end{bmatrix} \qquad \text{(IP)} \\ & x \geq 0 \text{ and integer} \end{aligned}$$



## Total Unimodular Matrices

### Resume'



**Recall: Totally Unimodular Matrices**

**Definition:** A matrix $A$ is totally unimodular (TU) if every square submatrix of $A$ has determinant +1, -1 or 0.

**Proposition 1:** The linear program $\max\{cx : Ax \leq b, x \in \mathbf{R}^m_+\}$ has an integral optimal solution for all integer vectors $b$ for which it has a finite optimal value if and only if $A$ is totally unimodular

Recognizing total unimodularity can be done in polynomial time (see [Schrijver, 1986])



## Total Unimodular Matrices

### Resume'



**Definition**

A $(0, 1)$–matrix $B$ has the consecutive 1's property if for any column $j$, $b_{ij} = b_{i'j} = 1$ with $i < i'$ implies $b_{lj} = 1$ for $i < l < i'$.
That is, if there is a permutation of the rows such that the 1's in each column appear consecutively.

Whether a matrix has the consecutive 1's property can be determined in polynomial time [ D. R. Fulkerson and O. A. Gross; Incidence matrices and interval graphs. 1965 Pacific J. Math. 15(3) 835-855.]

A matrix with consecutive 1's property is called an interval matrix

**Proposition:** Consecutive 1's matrices are TUM.

What about this matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

**Definition** A $(0,1)$-matrix $B$ has the circular 1's property for rows (resp. for columns) if the columns of $B$ can be permuted so that the 1's in each row are circular, that is, appear in a circularly consecutive fashion

The circular 1's property for columns does not imply circular 1's property for rows.

Whether a matrix has the circular 1's property for rows (resp. columns) can be determined in $O(m^2 n)$ time [A. Tucker, Matrix characterizations of circular-arc graphs. (1971) Pacific J. Math. 39(2) 535-545]





Integer programs where the constraint matrix $A$ have the circular 1's property for **rows** can be solved efficiently as follows:

Step 1 Solve the linear relaxation of (IP) to obtain $x'_1, \ldots, x'_n$. If $x'_1, \ldots, x'_n$ are integer, then it is optimal for (IP) and STOP. Otherwise go to Step 2.

Step 2 Form two linear programs LP1 and LP2 from the relaxation of the original problem by adding respectively the constraints

$$x_1 + \ldots + x_n = \lfloor x'_1 + \ldots + x'_n \rfloor \qquad \text{(LP1)}$$

and

$$x_1 + \ldots + x_n = \lceil x'_1 + \ldots + x'_n \rceil \qquad \text{(LP2)}$$

From LP1 and LP2 an integral solution certainly arises (P)



## Cyclic Staffing with Overtime

- Hourly requirements $b_i$
- Basic work shift 8 hours
- Overtime of up to additional 8 hours possible

minimize cx

subject to

$$\begin{array}{c} 07\\08\\09\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24\\01\\02\\03\\04\\05\\06 \end{array}
\begin{bmatrix}
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&1&1&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&1&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&0&1&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&0&0&1&1&1&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&1&1&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&1&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&1&1 \\
1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&1 \\
0&1&1&1&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&1&1&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&1&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&1&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1 & 1&1&1&1&1&1&1&1&1 & 0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0 & 0&1&1&1&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&1&1&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&0&1&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&1&1&1&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&1&1&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&1&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&1&1 & 1&1&1&1&1&1&1&1&1 \\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0&0&1 & 1&1&1&1&1&1&1&1&1
\end{bmatrix} x \geq b$$

$x \geq 0$ and integer.



## Days-Off Scheduling

- Guarantee two days-off each week, including every other weekend.

IP with matrix $A$:

$$\begin{array}{l} \\ \\ \text{first week} \\ \\ \\ \\ \\ \\ \\ \\ \text{second week} \\ \\ \\ \end{array}
\left[\begin{array}{cccccc|cccccc}
1&1&1&1&1&1 & 1&1&1&1&1&0 \\
1&1&1&1&1&1 & 1&1&1&1&0&0 \\
1&1&1&1&1&1 & 1&1&1&0&0&1 \\
1&1&1&1&1&1 & 1&1&0&0&1&1 \\
1&1&1&1&1&1 & 1&0&0&1&1&1 \\
0&0&0&0&0&0 & 0&0&1&1&1&1 \\
0&0&0&0&0&0 & 0&1&1&1&1&1 \\
\hline
0&1&1&1&1&1 & 1&1&1&1&1&1 \\
0&0&1&1&1&1 & 1&1&1&1&1&1 \\
1&0&0&1&1&1 & 1&1&1&1&1&1 \\
1&1&0&0&1&1 & 1&1&1&1&1&1 \\
1&1&1&0&0&1 & 1&1&1&1&1&1 \\
1&1&1&1&0&0 & 0&0&0&0&0&0 \\
1&1&1&1&1&0 & 0&0&0&0&0&0
\end{array}\right]$$

## Cyclic Staffing with Part-Time Workers

- Columns of $A$ describe the work-shifts
- Part-time employees can be hired for each time period $i$ at cost $c'_i$ per worker

$$\begin{aligned} \min \quad & cx + c'x' \\ st \quad & Ax + Ix' \geq b \\ & x, x' \geq 0 \text{ and integer} \end{aligned}$$





## Cyclic Staffing with Linear Penalties for Understaffing and Overstaffing

- demands are not rigid
- a cost $c'_i$ for understaffing and a cost $c''_i$ for overstaffing

$$\begin{aligned} \min \quad & cx + c'x' + c''(b - Ax - x') \\ st \quad & Ax + Ix' \geq b \\ & x, x' \geq 0 \text{ and integer} \end{aligned}$$



# Nurse Scheduling

## A CP approach



- Hospital: head nurses on duty seven days a week 24 hours a day
- Three 8 hours shifts per day (1: daytime, 2: evening, 3: night)
- In a day each shift must be staffed by a different nurse
- The schedule must be the same every week
- Four nurses are available (A,B,C,D) and must work at least 5 days a week.
- No shift should be staffed by more than two different nurses during the week
- No employee is asked to work different shifts on two consecutive days
- An employee that works shifts 2 and 3 must do so at least two days in a row.





Mainly a feasibility problem

## A CP approach

Two solution representations

| | Sun | Mon | Tue | Wed | Thu | Fri | Sat |
|---|---|---|---|---|---|---|---|
| Shift 1 | A | B | A | A | A | A | A |
| Shift 2 | C | C | C | B | B | B | B |
| Shift 3 | D | D | D | D | C | C | D |

| | Sun | Mon | Tue | Wed | Thu | Fri | Sat |
|---|---|---|---|---|---|---|---|
| Worker A | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| Worker B | 0 | 1 | 0 | 2 | 2 | 2 | 2 |
| Worker C | 2 | 2 | 2 | 0 | 3 | 3 | 0 |
| Worker D | 3 | 3 | 3 | 3 | 0 | 0 | 3 |

Variables: $w_{sd}$ nurse assigned to shift $s$ on day $d$
and $y_{id}$ the shift assigned to $i$ on day $d$

$$w_{sd} \in \{A, B, C, D\} \qquad y_{id} \in \{0, 1, 2, 3\}$$

Three different nurses are scheduled each day

$$\texttt{alldiff}(w_{\cdot d}) \qquad \forall d$$

Every nurse is assigned to at least 5 days of work

$$\texttt{cardinality}(w_{\cdot\cdot} \mid (A, B, C, D), (5, 5, 5, 5), (6, 6, 6, 6))$$

At most two nurses work any given shift

$$\texttt{nvalues}(w_{s\cdot} \mid 1, 2) \qquad \forall s$$





All shifts assigned for each day

$$\texttt{alldiff}(y_{\cdot d}) \qquad \forall d$$

Maximal sequence of consecutive variables that take the same values

$$\texttt{stretch-cycle}(y_{i\cdot} \mid (2, 3), (2, 2), (6, 6), P)$$
$$\forall i,\ P = \{(s, 0), (0, s) \mid s = 1, 2, 3\}$$

Channeling constraints between the two representations:
on any day, the nurse assigned to the shift to which nurse $i$ is assigned must be nurse $i$ (element constraint)

$$w_{y_{id}, d} = i \qquad \forall i, d$$
$$y_{w_{sd}, d} = s \qquad \forall s, d$$





The complete CP model

$$\text{Alldiff:} \left\{ \begin{array}{c} (w_{\cdot d}) \\ (y_{\cdot d}) \end{array} \right\}, \text{ all } d$$

$$\text{Cardinality: } (w_{\cdot\cdot} \mid (A, B, C, D), (5, 5, 5, 5), (6, 6, 6, 6))$$

$$\text{Nvalues: } (w_{s\cdot} \mid 1, 2), \text{ all } s$$

$$\text{Stretch-cycle: } (y_{i\cdot} \mid (2, 3), (2, 2), (6, 6), P), \text{ all } i$$

$$\text{Linear:} \left\{ \begin{array}{l} w_{y_{id} d} = i, \text{ all } i \\ y_{w_{sd} d} = s, \text{ all } s \end{array} \right\}, \text{ all } d$$

$$\text{Domains:} \left\{ \begin{array}{l} w_{sd} \in \{A, B, C, D\},\ s = 1, 2, 3 \\ y_{id} \in \{0, 1, 2, 3\},\ i = A, B, C, D \end{array} \right\}, \text{ all } d$$





Constraint Propagation:

- alldiff: matching
- nvalues: max flow
- stretch: poly-time dynamic programming
- index expressions $w_{y_{id} d}$ replaced by $z$ and constraint: $\texttt{element}(y, x, z)$: $z$ be equal to $y$-th variable in list $x_1, \ldots, x_m$

Search:

- branching by splitting domanins with more than one element
- first fail branching
- symmetry breaking:
  - employees are indistinguishable
  - shifts 2 and 3 are indistinguishable
  - days can be rotated

  Eg: fix $A, B, C$ to work $1, 2, 3$ resp. on sunday

# Heuristic Methods

- Local search and metaheuristic methods are used if the problem has large scale.
- Procedures are very similar to what we saw for course timetabling.