DM204, 2010
SCHEDULING, TIMETABLING AND ROUTING

Lecture 3

Resource Constrained Project Scheduling
Linear and Integer Programming (1)

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Outline

Outline

RCPSP

Resource Constrained Project Scheduling Model

Given:

- activities (jobs) $j = 1, \ldots, n$
- renewable resources $i = 1, \ldots, m$
- amount of resources available R_i
- processing times p_j
- amount of resource used r_{ij}
- precedence constraints $j \rightarrow k$

Further generalizations

- Time dependent resource profile $R_i(t)$
  given by (t_i^μ, R_i^μ) where $0 = t_i^1 < t_i^2 < \ldots < t_i^{m_i} = T$
  Disjunctive resource, if $R_k(t) = \{0, 1\}$; cumulative resource, otherwise
- Multiple modes for an activity j
  processing time and use of resource depends on its mode m: p_{jm}, r_{jkm}.

Modeling

Assignment 1

- A contractor has to complete n activities.
- The duration of activity j is p_j
- each activity requires a crew of size W_j.
- The activities are not subject to precedence constraints.
- The contractor has W workers at his disposal
- his objective is to complete all n activities in minimum time.

Assignment 2

- Exams in a college may have different duration.
- The exams have to be held in a gym with W seats.
- The enrollment in course j is W_j and
- all W_j students have to take the exam at the same time.
- The goal is to develop a timetable that schedules all n exams in minimum time.
- Consider both the cases in which each student has to attend a single exam as well as the situation in which a student can attend more than one exam.

Assignment 3

- In a basic high-school timetabling problem we are given m classes $c_1, \ldots, c_m$,
- h teachers $a_1, \ldots, a_h$ and
- T teaching periods $t_1, \ldots, t_T$.
- Furthermore, we have lectures $i = l_1, \ldots, l_n$.
- Associated with each lecture is a unique teacher and a unique class.
- A teacher a_j may be available only in certain teaching periods.
- The corresponding timetabling problem is to assign the lectures to the teaching periods such that
  - each class has at most one lecture in any time period
  - each teacher has at most one lecture in any time period,
  - each teacher has only to teach in time periods where he is available.

Assignment 4

- A set of jobs $J_1, \ldots, J_g$ are to be processed by auditors $A_1, \ldots, A_m$.
- Job J_l consists of n_l tasks ($l = 1, \ldots, g$).
- There are precedence constraints $i_1 \to i_2$ between tasks i_1, i_2 of the same job.
- Each job J_l has a release time r_l, a due date d_l and a weight w_l.
- Each task must be processed by exactly one auditor. If task i is processed by auditor A_k, then its processing time is p_{ik}.
- Auditor A_k is available during disjoint time intervals $[s_k^\nu, l_k^\nu]$ ($\nu = 1, \ldots, m$) with $l_k^\nu < s_k^\nu$ for $\nu = 1, \ldots, m_k - 1$.
- Furthermore, the total working time of A_k is bounded from below by H_k^- and from above by H_k^+ with $H_k^- \leq H_k^+$ ($k = 1, \ldots, m$).
- We have to find an assignment $\alpha(i)$ for each task $i = 1, \ldots, n := \sum_{l=1}^{g} n_l$ to an auditor $A_{\alpha(i)}$ such that
  - each task is processed without preemption in a time window of the assigned auditor
  - the total workload of A_k is bounded by H_k^- and H_k^k for $k = 1, \ldots, m$.
  - the precedence constraints are satisfied,
  - all tasks of J_l do not start before time r_l, and
  - the total weighted tardiness $\sum_{l=1}^{g} w_l T_l$ is minimized.

Outline

Mathematical Programming

Linear, Integer, Nonlinear

program = optimization problem

$$\begin{aligned} \min \quad & f(x) \\ & g_i(x) = 0, \quad i = 1, 2, \ldots, k \\ & h_j(x) \leq 0, \quad j = 1, 2, \ldots, m \\ & x \in \mathbf{R}^n \end{aligned}$$

general (nonlinear) program (NLP)

$$\begin{aligned} \min \quad & c^T x \\ & Ax = a \\ & Bx \leq b \\ & x \geq 0 \\ & (x \in \mathbf{R}^n) \end{aligned}$$

linear program (LP)

$$\begin{aligned} \min \quad & c^T x \\ & Ax = a \\ & Bx \leq b \\ & x \geq 0 \\ & (x \in \mathbf{Z}^n) \\ & (x \in \{0, 1\}^n) \end{aligned}$$

integer (linear) program (IP, MIP)

Linear Programming

Linear Program in standard form

$$\begin{aligned} \min \quad & c_1 x_1 + c_2 x_2 + \ldots c_n x_n \\ s.t. \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\ & a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\ & \vdots \\ & a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_n \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Historic Roots

- 1939 L. V. Kantorovitch: Foundations of linear programming (Nobel Prize 1975)
- George J. Stigler's 1945 (Nobel Prize 1982) "Diet Problem": "the first linear program"
  find the cheapest combination of foods that will
  satisfy the daily requirements of a person
  Army's problem had 77 unknowns and 9 constraints.
  http://www.mcs.anl.gov/home/otc/Guide/CaseStudies/diet/index.html
- 1947 G.B. Dantzig: Invention of the simplex algorithm
- Founding fathers:
  - 1950s Dantzig: Linear Programming 1954, the Beginning of IP
    G. Dantzig, D.R. Fulkerson, S. Johnson TSP with 49 cities
  - 1960s Gomory: Integer Programming

LP Theory

- Max-Flow Min-Cut Theorem
  The maximal (s,t)-flow in a capaciatetd network is equal to the minimal capacity of an (s,t)-cut

- The Duality Theorem of Linear Programming

$$\begin{array}{ll} \max & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \min & y^T b \\ & y^T A \geq c^T \\ & y \geq 0 \end{array}$$

  If feasible solutions to both the primal and the dual problem in a pair of dual LP problems exist, then there is an optimum solution to both systems and the optimal values are equal.

LP Theory

- Max-Flow Min-Cut Theorem
  does not hold if several source-sink relations are given (multicommodity flow)

- The Duality Theorem of Integer Programming

$$\begin{array}{ll} \max & c^T x \\ & Ax \leq b \\ & x \geq 0 \\ & x \in \mathbf{Z}^n \end{array} \qquad \leq \qquad \begin{array}{ll} \min & y^T b \\ & y^T A \geq c^T \\ & y \geq 0 \\ & y \in \mathbf{Z}^n \end{array}$$

LP Solvability

- Linear programs can be solved in polynomial time with
  the Ellipsoid Method (Khachiyan, 1979)
  Interior Point Methods (Karmarkar, 1984, and others)
- Open: is there a strongly polynomial time algorithm for the solution of LPs?
- Certain variants of the Simplex Algorithm run - under certain conditions - in expected polynomial time (Borgwardt, 1977...)
- Open: Is there a polynomial time variant of the Simplex Algorithm?

LP Solvability

Theorem (Grötschel, Lovász, Schrijver 1979, 1988))

(modulo technical details)
There exists a polynomial time algorithm to minimize convex functions (e.g., linear functions) over the elements of a class of convex bodies $\mathbf{K}$ *(e. g. polyhedra) if and only if, there exists a polynomial time algorithm that decides, for any given point* x*, whether* x *is in* $\mathbf{K}$*, and that, when* x *is not in* $\mathbf{K}$*, finds a hyperplane that separates* x *from* $\mathbf{K}$*.*

Short version: Optimization and Separation are polynomial-time equivalent.

Consequence: Theoretical Foundation of cutting plane algorithms.

IP Solvability

- Theorem
  Integer, 0/1, and mixed integer programming are NP-hard.
- Consequence
  - investigate special cases
  - special purpose algorithms
  - heuristics

Solution Algorithms

- Algorithms for the solution of nonlinear programs
- Algorithms for the solution of linear programs
  - 1) Fourier-Motzkin Elimination (hopeless)
  - 2) The Simplex Method (good, above all with duality)
  - 3) The Ellipsoid Method (total failure)
  - 4) Interior-Point/Barrier Methods (good)
- Algorithms for the solution of integer programs
  - 1) Branch & Bound
  - 2) Cutting Planes

Nonlinear programming

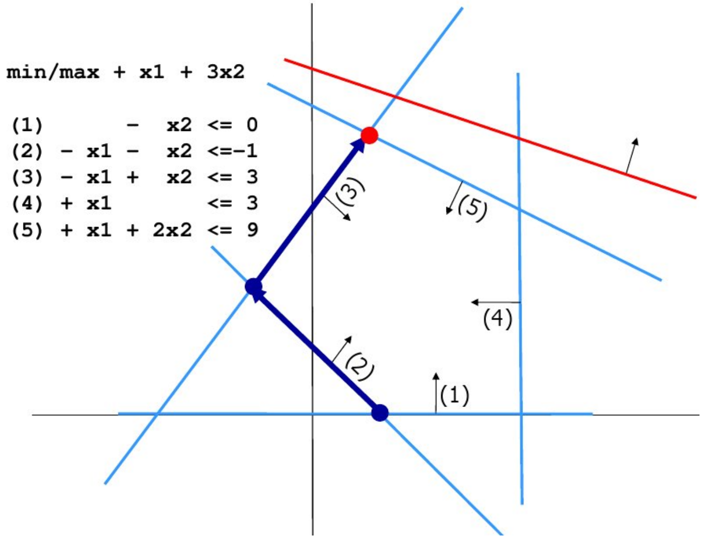
- Iterative methods that solve the equation and inequality systems representing the necessary local optimality conditions (Kuhn-Tucker sufficient conditions)
- Steepest descent
- Newton method
- Subgradient method (for piecewise functions)

Linear programming

The Simplex Method

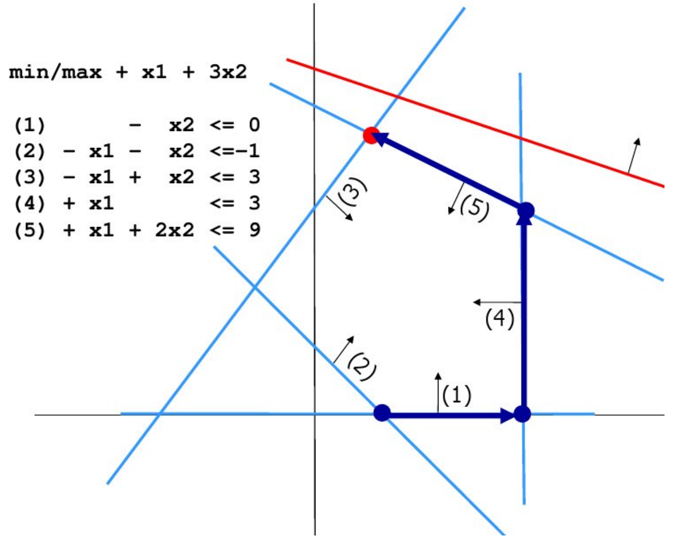
- Dantzig, 1947: primal Simplex Method
- Lemke, 1954; Beale, 1954: dual Simplex Method
- Dantzig, 1953: revised Simplex Method
- ....
- Underlying Idea: Find a vertex of the set of feasible LP solutions (polyhedron) and move to a better neighbouring vertex, if possible.

The simplex method



The simplex method

Scheduling
Math Programming
Introduction
Solution Algorithms



The simplex method

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Hirsch Conjecture
If P is a polytope of dimension n with m facets then every vertex of P can be reached from any other vertex of P on a path of length at most $m - n$.

In the example before: $m = 5$, $n = 2$ and $m - n = 3$, conjecture is true.

At present, not even a polynomial bound on the path length is known.
Best upper bound: Kalai, Kleitman (1992): The diameter of the graph of an n-dimensional polyhedron with m facets is at most $m^{(log n+1)}$.
Lower bound: Holt, Klee (1997): at least $m - n$ (m, n large enough).

Integer Programming (easy)

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special „simple" combinatorial optimization problems Finding a:

- minimum spanning tree
- shortest path
- maximum matching
- maximal flow through a network
- cost-minimal flow
- ...

solvable in polynomial time by special purpose algorithms

Integer Programming (hard)

special „hard" combinatorial optimization problems

- traveling salesman problem
- location and routing
- set-packing, partitioning, -covering
- max-cut
- linear ordering
- scheduling (with a few exceptions)
- node and edge colouring
- ...

NP-hard (in the sense of complexity theory)
The most successful solution techniques employ linear programming.

Integer Programming (hard)

- 1) Branch & Bound
- 2) Cutting Planes

Branch & cut, Branch & Price (column generation), Branch & Cut & Price

Summary

- We can solve today explicit LPs with
  - up to 500,000 of variables and
  - up to 5,000,000 of constraints

  in relatively short running times.
- We can solve today structured implicit LPs (employing column generation and cutting plane techniques) in special cases with hundreds of million (and more) variables and almost infinitely many constraints in acceptable running times. (Examples: TSP, bus circulation in Berlin)

[Martin Grötschel, Block Course at TU Berlin,
"Combinatorial Optimization at Work", 2005
http://co-at-work.zib.de/berlin/]