

DM204, 2010
SCHEDULING, TIMETABLING AND ROUTING

Lecture 6
Mixed Integer Programming
Network Flows

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Min cost flow problem

Min cost flow problem is a central model in network flows:

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij} \quad (1)$$

$$\text{s.t.} \quad \sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = b_i, \quad \text{for all } i \in N \quad (2)$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in A \quad (3)$$

$$x_{jt} \in \mathbf{R} \quad (4)$$

(2) mass balance constrains $Nx = b$

(3) flow bound constraint $l \leq x \leq u$

Course Overview

- ✓ Problem Introduction
 - ✓ Scheduling classification
 - ✓ Scheduling complexity
 - ✓ RCPSP
- General Methods
 - Integer Programming
 - Constraint Programming
 - Heuristics
 - Dynamic Programming and Branch and Bound
- Scheduling
 - Single Machine
 - Parallel Machine and Flow Shop Models
 - Job Shop
 - Resource Constrained Project Scheduling Model
- Timetabling
 - Reservations and Education
 - University Timetabling
 - Crew Scheduling
 - Public Transports
- Vehicle Routing
 - Capacited Models
 - Time Windows models
 - Rich Models

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2

Min Cost Flow Models

- Shortest path
- Max flow \equiv min cut
- Assignment
- Transportation
- Network transformations: node splitting

Simplex: Recap.

Simplex in matrix form

$$\min \{cx \mid Ax = b, x \geq\}$$

In matrix form:

$$\begin{bmatrix} 0 & A \\ -1 & c \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- $B = \{1, 2, \dots, p\}$ basic variables
- $\mathcal{L} = \{1, 2, \dots, q\}$ non-basis variables (will be set to lower bound = 0)
- (B, \mathcal{L}) basis structure
- $x_B, x_{\mathcal{L}}, c_B, c_{\mathcal{L}},$
- $B = [A_1, A_2, \dots, A_p], L = [A_{p+1}, A_{p+2}, \dots, A_{p+q}]$

$$\begin{bmatrix} 0 & B & L \\ -1 & c_B & c_{\mathcal{L}} \end{bmatrix} \begin{bmatrix} z \\ x_B \\ x_{\mathcal{L}} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$Bx_B + Lx_{\mathcal{L}} = b \Rightarrow x_B + B^{-1}Lx_{\mathcal{L}} = B^{-1}b \Rightarrow \begin{cases} x_{\mathcal{L}} = 0 \\ x_B = B^{-1}b \end{cases}$$

- When is $x_B = B^{-1}b$ integer? When $\det(B) = \pm 1$
- Totally unimodular matrices (TUM)
- Node-arc incidence matrix for directed graph are TUM
- Node-arc incidence matrix for undirected graph are TUM if they do not contain odd cycles
- Matching problems: bipartite/nonbipartite and cardinality/weighted

Multi commodity flows

commodities that share arc capacity

$$\min \sum_{1 \leq k \leq K} c_{ij}^k x_{ij}^k \tag{1}$$

$$\text{s.t. } \sum_{1 \leq k \leq K} x_{ij}^k \leq u_{ij} \forall (i, j) \in A \tag{2}$$

$$Nx^k = b^k \quad \text{for all } i \in N \tag{3}$$

$$0 \leq x_{ij}^k \leq u_{ij} \quad \forall (i, j) \in A \tag{4}$$

$$\tag{5}$$

(2) bundle constraints

(3) mass balance constraints $Nx = b$