DM204, 2010
SCHEDULING, TIMETABLING AND ROUTING

Lecture 6
Mixed Integer Programming
Netowrk Flows

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Course Overview

- ✔ Problem Introduction
  - ✔ Scheduling classification
  - ✔ Scheduling complexity
  - ✔ RCPSP
- General Methods
  - Integer Programming
  - Constraint Programming
  - Heuristics
  - Dynamic Programming and Branch and Bound
- Scheduling
  - Single Machine
  - Parallel Machine and Flow Shop Models
  - Job Shop
  - Resource Constrained Project Scheduling Model
- Timetabling
  - Reservations and Education
  - University Timetabling
  - Crew Scheduling
  - Public Transports
- Vechicle Routing
  - Capacited Models
  - Time Windows models
  - Rich Models

Min cost flow problem

Min cost flow problem is a central model in network flows:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{1}$$

$$\text{s.t.} \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i, \quad \text{for all } i \in N \tag{2}$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in A \tag{3}$$

$$x_{jt} \in \mathbf{R} \tag{4}$$

(2) mass balance constrains $Nx = b$
(3) flow bound constraint $l \leq x \leq u$

Min Cost Flow Models

- Shortest path
- Max flow $\equiv$ min cut
- Assignment
- Transportation
- Network transformations: node splitting

Simplex: Recap.

Simplex in matrix form

$$\min \{cx \mid Ax = b, x \geq\}$$

In matrix form:

$$\begin{bmatrix} 0 & A \\ -1 & c \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- $\mathcal{B} = \{1, 2, \ldots, p\}$ basic variables
- $\mathcal{L} = \{1, 2, \ldots, q\}$ non-basis variables (will be set to lower bound = 0)
- $(\mathcal{B}, \mathcal{L})$ basis structure
- $x_\mathcal{B}, x_\mathcal{L}, c_\mathcal{B}, c_\mathcal{L}$,
- $B = [A_1, A_2, \ldots, A_p]$, $L = [A_{p+1}, A_{p+2}, \ldots, A_{p+q}]$

$$\begin{bmatrix} 0 & B & L \\ -1 & c_\mathcal{B} & c_\mathcal{L} \end{bmatrix} \begin{bmatrix} z \\ x_\mathcal{B} \\ x_\mathcal{L} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$B x_{\mathcal{B}} + L x_{\mathcal{L}} = b \quad \Rightarrow \quad x_{\mathcal{B}} + B^{-1} L x_{\mathcal{L}} = B^{-1} b \quad \Rightarrow \quad \left[\begin{array}{l} x_{\mathcal{L}} = 0 \\ x_{\mathcal{B}} = B^{-1} b \end{array} \right.$$

- When is $x_{\mathcal{B}} = B^{-1} b$ integer? When $\det(B) = \pm 1$
- Totally unimodular matrices (TUM)
- Node-arc incidence matrix for directed graph are TUM
- Node-arc incidence matrix for undirected graph are TUM if they do not contain odd cycles
- Matching problems: bipartite/nonbipartite and cardinality/weighted

Multi commodity flows

commodities that share arc capacity

$$\min \sum_{1 \leq k \leq K} c_{ij}^k x_{ij}^k \tag{1}$$

$$\text{s.t.} \sum_{1 \leq k \leq K} x_{ij}^k \leq u_{ij} \forall (i,j) \in A \tag{2}$$

$$N x^k = b^k \quad \text{for all } i \in N \tag{3}$$

$$0 \leq x_{jt}^k \leq u_{ij}^k \quad \forall (i,j) \in A \tag{4}$$

$$\tag{5}$$

(2) bundle constraints
(3) mass balance constraints $Nx = b$