

## DM545/DM871 – Linear and integer programming

### Sheet 2, Spring 2020 [pdf format]

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#### Solution:

Included.

**Exercise 1** List all vertices of the polyhedron  $Ax \leq b$  characterized by the following matrices  $A$  and  $b$ :

$$A = \begin{bmatrix} 2 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

You are encouraged to use Python for carrying out the calculations.

#### Solution:

The polyhedron in standard form is defined by  $Ax \leq b$ . Let  $\bar{A}$  be the matrix after the introduction of the slack variables, ie,

$$\bar{A} = \begin{bmatrix} 2 & 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let  $m$  be the number of constraints and  $n$  the number of variables of the system of linear equations  $\bar{A}\bar{x} = b$ . Hence,  $\bar{A} \in \mathbb{R}^{m \times n}$ ,  $\bar{x} \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ . Recall that the system admits solutions if and only if:  $\text{rank}(\bar{A}) = \text{rank}(\bar{A} | b)$ .

If  $n < m$  and hence  $\text{rank}(\bar{A}) \leq n$  the system is overdetermined and likely infeasible, unless  $\text{rank}(\bar{A}) = \text{rank}(\bar{A} | b)$ , which would imply that the constraints are linearly dependent. Linearly dependent constraints can be removed as they are redundant. Thus, we can assume that  $n \geq m$  and most likely  $n > m$  since we introduced a slack variable for each constraint. Moreover, since we removed linearly dependent constraints:  $\text{rank}(\bar{A}) = m$  and consequently it must be  $\text{rank}(\bar{A}) = \text{rank}(\bar{A} | b)$ . Under the assumption that  $n > m$  then the system is underdetermined. The solutions have  $n - m$  free variables and the solution space has dimension  $n - m$ . It is the simplex represented by the intersection of  $\bar{A}\bar{x} = b$  and  $\bar{x} \geq 0$ . The vertices of this simplex in  $\mathbb{R}^n$  are the vertices of the polyhedron described by  $A$  and  $b$ . Algebraically, they correspond to the basic feasible solutions of the linear system  $\bar{A}\bar{x} = b$ . This means that to find the vertices of the polyhedron we need to enumerate all basic solutions of  $\bar{A}\bar{x} = b$ . We recall that a basic solution is given by a subset  $B$  of size  $m$  of the indices of columns of the matrix  $\bar{A}$  and is such that  $A_B$ , the so-called basis matrix, is non-singular and  $\bar{x}_B = A_B^{-1}b \geq 0$  and  $\bar{x}_N = 0$ . To determine all bases we need to generate all combinations of size  $m = 3$  of the  $n = 7$  columns. We write a python script to do the calculations for us:

```
import scipy as sc
import scipy.linalg as sl
import sympy as sy
import itertools as it

A = sc.array([[2,0,1,-4],[0,1,0,2],[0,0,1,0]])
I = sc.identity(3)
A = sc.concatenate([A,I],axis=1)
print A
b = sc.array([3,1,1])

for e in it.combinations(range(7),3):
    if sl.det(A[:,e]) != 0:
        x = sc.dot(sl.inv(A[:,e]),b)
        if (x>=0).all:
```

```

    print e,x
    else:
        print e," infesible"
    else:
        print e, sy.Matrix(sc.column_stack([A[:,e],b])).rref()

```

## Exercise 2\* Simplex method

This is part of the first exercise (Opgave 1) in the written exam of 2008.  
Consider the following linear programming problem (P1)

$$\begin{aligned}
 & \text{maximize} && 2x_1 + 4x_2 - x_3 \\
 & \text{subject to} && 2x_1 - x_3 \leq 6 \\
 & && 3x_2 - x_3 \leq 9 \\
 & && x_1 + x_2 \leq 4 \\
 & && x_1, x_2, x_3 \geq 0
 \end{aligned}$$

- Rewrite the problem in equational standard form adding the slack variables  $x_4, x_5, x_6$  to the three constraints above, respectively, and write the first simplex tableau with  $x_4, x_5, x_6$  as basic solution.

**Solution:**

We write the initial tableaux:

$$\begin{bmatrix}
 2 & 0 & -1 & 1 & 0 & 0 & 0 & 6 \\
 0 & 3 & -1 & 0 & 1 & 0 & 0 & 9 \\
 1 & 1 & 0 & 0 & 0 & 1 & 0 & 4 \\
 2 & 4 & -1 & 0 & 0 & 0 & 1 & 0
 \end{bmatrix}$$

Or also:

	x1	x2	x3	x4	x5	x6	-z	b
	2	0	-1	1	0	0	0	6
	0	3	-1	0	1	0	0	9
	1	1	0	0	0	1	0	4
	2	4	-1	0	0	0	1	0

- Argue that  $x_2$  can be brought in the basis with advantage and perform one pivot iteration that brings  $x_2$  into the basic solution.

**Solution:**

$x_2$  has positive reduced cost, hence worth bringing up.

pivot column: 2 pivot row: 3 pivot: 3

$$\begin{bmatrix}
 2 & 0 & -1 & 1 & 0 & 0 & 0 & 6 \\
 0 & 1 & -1/3 & 0 & 1/3 & 0 & 0 & 3 \\
 1 & 0 & 1/3 & 0 & -1/3 & 1 & 0 & 1 \\
 2 & 0 & 1/3 & 0 & -4/3 & 0 & 1 & -12
 \end{bmatrix}$$

- After another pivot iteration, it is  $x_1$  that can be brought with advantage in the basis (you do not have to perform this iteration), reaching the following simplex tableau:

x1	x2	x3	x4	x5	x6	-z	b
0	0	-5/3	1	2/3	-2	0	4
0	1	-1/3	0	1/3	0	0	3
1	0	1/3	0	-1/3	1	0	1
0	0	-1/3	0	-2/3	-2	1	-14

Argue that an optimal solution is found and give the solution together with its objective value.

### Solution:

The optimal solution is found because all reduced costs are non-positive. The objective function value is 14.

We show here for completeness all iterations of the simplex from the first tableau above:

pivot column: 2

pivot row: 2

pivot: 3

x1	x2	x3	x4	x5	x6	-z	b
2	0	-1	1	0	0	0	6
0	1	-1/3	0	1/3	0	0	3
1	0	1/3	0	-1/3	1	0	1
2	0	1/3	0	-4/3	0	1	-12

pivot column: 1

pivot row: 3

pivot: 1

x1	x2	x3	x4	x5	x6	-z	b
0	0	-5/3	1	2/3	-2	0	4
0	1	-1/3	0	1/3	0	0	3
1	0	1/3	0	-1/3	1	0	1
0	0	-1/3	0	-2/3	-2	1	-14

### Exercise 3\* Simplex method

Solve the following LP problem carrying out the simplex operations:

$$\begin{aligned}
 &\text{maximize} && 5x_1 + 4x_2 + 3x_3 \\
 &\text{subject to} && 2x_1 + 3x_2 + x_3 \leq 5 \\
 &&& 4x_1 + x_2 + 2x_3 \leq 11 \\
 &&& 3x_1 + 4x_2 + 2x_3 \leq 8 \\
 &&& x_1, x_2, x_3 \geq 0
 \end{aligned}$$

You are free to use any of the two representations, tableau or dictionary.

You can also get help from Python. You find a tutorial for what you need at this link:

<http://www.imada.sdu.dk/~marco/DM545/Resources/Ipython/Tutorial4Exam.html>.

**Solution:**

```
%run utils
A=array([[2,f(3,1),1,1,0,0,0,5],[4,1,2,0,1,0,0,11],[3,4,2,0,0,1,0,8],[5,4,3,0,0,0,1,0]])
tableau(A)
# enough that one is a fraction to make all matrix of type fraction
# First simplex iteration
A[0,:] = f(1,2)*A[0,:]
A[1,:] = A[1,:]-f(4,1)*A[0,:]
A[2,:] = A[2,:]-f(3,1)*A[0,:]
A[3,:] = A[3,:]-f(5,1)*A[0,:]
tableau(A)
```

x1	x2	x3	x4	x5	x6	-z	b
2	3	1	1	0	0	0	5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
5	4	3	0	0	0	1	0

pivot column: 1  
 pivot row: 1  
 pivot: 2

x1	x2	x3	x4	x5	x6	-z	b
1	3/2	1/2	1/2	0	0	0	5/2
0	-5	0	-2	1	0	0	1
0	-1/2	1/2	-3/2	0	1	0	1/2
0	-7/2	1/2	-5/2	0	0	1	-25/2

pivot column: 3  
 pivot row: 3  
 pivot: 1/2

x1	x2	x3	x4	x5	x6	-z	b
1	2	0	2	0	-1	0	2
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	-3	0	-1	0	-1	1	-13

**Exercise 4**

Solve the following linear programming problem applying the simplex algorithm:

Enter the linear programming problem here:

Maximize  
 Minimize  
 Show only the region defined by the following constraints:

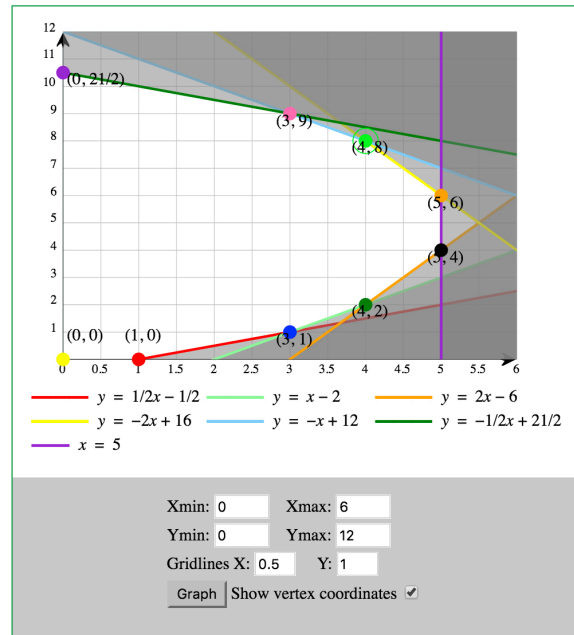
$z = 3x + 2y$  subject to the constraints:

$x - 2y \leq 1$   
 $x - y \leq 2$   
 $2x - y \leq 6$   
 $x \leq 5$   
 $2x + y \leq 16$   
 $x + y \leq 12$   
 $x + 2y \leq 21$

Rounding: 4 decimal places Fraction Mode

The solution will appear below.

	$z = 3x + 2y = v$	
● (5, 4)	$2x - y = 6$ $x = 5$	23
● (5, 6)	$x = 5$ $2x + y = 16$	27
● (4, 8)	$2x + y = 16$ $x + y = 12$	28 Maximum
● (3, 9)	$x + y = 12$ $x + 2y = 21$	27
● (0, 21/2)	$x + 2y = 21$ $x = 0$	21
● (0, 0)	$x = 0$ $y = 0$	0



$$\begin{aligned}
 &\text{maximize} && 3x_1 + 2x_2 \\
 &\text{subject to} && x_1 - 2x_2 \leq 1 \\
 &&& x_1 - x_2 \leq 2 \\
 &&& 2x_1 - x_2 \leq 6 \\
 &&& x_1 \leq 5 \\
 &&& 2x_1 + x_2 \leq 16 \\
 &&& x_1 + x_2 \leq 12 \\
 &&& x_1 + 2x_2 \leq 21 \\
 &&& x_2 \leq 10 \\
 &&& x_1, x_2 \geq 0.
 \end{aligned}$$

[Hint: you can plot the feasibility region with one of the tools linked at the course web page: “Tools” -> “Web applications on the simplex” -> “LP Simplex” and use the clairvoyant’s rule to minimize the number of operations to carry out.]

**Solution:**

The initial solution of the simplex is at [0,0]. Then we can follow one of the two paths. The clairvoyant rule says that we should choose the direction that minimizes the path. This can be achieved by taking  $x_2$ . This yields a path of 3 arcs. Taking instead  $x_1$  in the basis at the beginning would lead to a path of length 6.

**Exercise 5\* Project Scheduling**

This exercise is a part of one that appeared in Exam 2011.

A small project has 6 sub-activities A, B, C, D, E, F whose individual dependency (shown by the immediate predecessors) is given in Figure 1. Here we also list the normal time (in weeks), the absolute minimum time and the cost of shortening the activity by one week.

The goal is to shorten the duration of the project to 19 weeks. This means that the duration of one or more activities has to be shortened. Of course we want to select these so that the total cost of shortening the duration to 19 weeks is minimized. Formulate this problem as a linear programming problem and argue that the optimal solution to this LP will provide the correct answer. Note that you must use the actual data in the LP formulation!

**Solution:**

We will use a variable  $x_i$  to indicate how much we will shorten activity  $i$  and another set of variables  $y_i$  which will indicate the earliest starting time of activity  $i$ . For each arc  $i \rightarrow j$  in the project network we

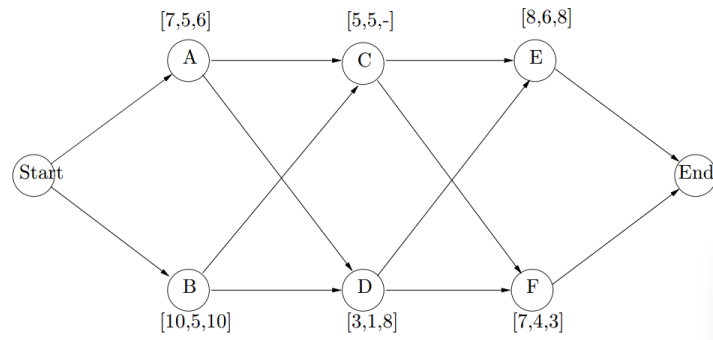


Figure 1: A network with activities on nodes for a small project with 6 activities. For each activity the following data is given in that order from left to right: normal time, minimum time in weeks, and the cost of shortening the duration of the activity by one week.

will add the constraint  $y_j \leq y_i + (d_i - x_i)$ . We also use a variable  $y_{end}$  to express that the dummy activity "end" cannot start before all its immediate predecessors have finished. Finally we add the constraint  $y_{end} \leq 19$  to force the total project time to be less or equal than 19.

$$\begin{aligned}
 & \min 6x_A + 10x_B + 8x_D + 8x_E + 3x_F \\
 & \text{subject to } y_C \geq y_A + (7 - x_A) \\
 & \quad y_C \geq y_B + (10 - x_B) \\
 & \quad y_D \geq y_A + (7 - x_B) \\
 & \quad y_D \geq y_B + (10 - x_B) \\
 & \quad y_E \geq y_C + 5 \\
 & \quad y_E \geq y_D + (3 - x_D) \\
 & \quad y_F \geq y_C + 5 \\
 & \quad y_F \geq y_D + (3 - x_D) \\
 & \quad y_{end} \geq y_E + (8 - x_E) \\
 & \quad y_{end} \geq y_F + (7 - x_F) \\
 & \quad y_{end} \geq 19 \\
 & \quad x_A \leq 2 \\
 & \quad x_B \leq 5 \\
 & \quad x_C \leq 2 \\
 & \quad x_D \leq 2 \\
 & \quad x_E \leq 3 \\
 & \quad x_F \leq 2 \\
 & \quad x_A, x_B, x_C, x_D, x_E, x_F \geq 0 \\
 & \quad y_A, y_B, y_C, y_D, y_E, y_F, y_{end} \geq 0
 \end{aligned}$$

The optimal solution to this LP will tell us to shorten activity  $i$  by  $x_i \geq 0$  units and since the cost we apply to each  $x_i$  is the per unit shortening cost of that activity, the cost of the solution will be that of shortening the project in the way suggested by the  $x_i$ 's. Conversely, any feasible shortening of projects corresponds to a solution to this LP whose cost (in the LP) is the actual cost of shortening the activities in the way suggested.

## Exercise 6 Quizzes

1. In 4D, how many hyperplanes need to intersect to give a point?

**Solution:**

4

2. In 4D, can a point be described by more than 4 hyperplanes?

**Solution:**

Yes, just think of a pyramid in 3D

3. Consider the intersection of  $n$  hyperplanes in  $n$  dimensions: when does it uniquely identify a point?

**Solution:**

when the rank of the matrix  $A$  of the linear system is  $n$  (or  $A$  is nonsingular)

Vertices of Polyhedra:

Consider the polyhedron described by  $Ax \leq \mathbf{b}$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $\mathbf{x} \in \mathbb{R}^n$ , that is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

4. For a point  $\mathbf{x}$  of a polyhedron, we define as *active* constraints those that are satisfied to equality by  $\mathbf{x}$ . How many constraints are *active* in a *vertex* of a polyhedron  $Ax \leq \mathbf{b}$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $\mathbf{x} \in \mathbb{R}^n$ ?

**Solution:**

at least  $n$ , rank of matrix of active constraints is  $n$

5. Does every point  $\mathbf{x}$  that activates  $n$  constraints form a vertex of the polyhedron?

**Solution:**

no, some may be not feasible, ie, intersection in a point outside of the polyhedron

6. Can a vertex activate more than  $n$  constraints?

**Solution:**

Yes, just look at the pyramid in 3 dim. However, the rank of the matrix of active constraints is still  $n$

7. What if there are more variables than constraints? If  $m > n$  then we can find a subset and then activate but what if  $m < n$ , can we have a vertex?

**Solution:**

No. In LP we deal with this issue by adding slack variables, they make us choose arbitrarily a vertex

8. Combinatorial explosion of vertices: how many constraints and vertices has an  $n$ -dimensional hypercube?

**Solution:**

To define a cube we need 6 constraints and there are  $2^3$  vertices. For an  $n$ -hypercube we need  $2n$  constraints and there are  $2^n$  vertices

9. If there are  $m$  constraints and  $n$  variables,  $m > n$ , what is an upper bound to the number of vertices?

**Solution:**

the number of possible active constraints is  $\binom{m}{n}$  it is an upper bound because:

- some combinations of constraints will not define a vertex, ie, if rows of matrix not independent
- some vertices are outside the polyhedron
- some vertices may activate more than  $n$  constraints and hence the same vertex can be given by more than  $n$  constraints

Tableaux and Vertices

10. For each of these three statements, say if they are true or false:

- One tableau  $\implies$  one vertex of the feasible region
- One tableau  $\longleftarrow$  one vertex of the feasible region
- One tableau  $\iff$  one vertex of the feasible region

**Solution:**

One tableau  $\longleftarrow$  one vertex of the feasible region degenerate vertices have several tableau associated

11. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{ll} \max & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & -z & b \\ \hline x_2 & 0 & 1 & 1/5 & -1/4 & 0 & 2 \\ x_1 & 1 & 0 & -1/5 & 1/2 & 0 & 8 \\ \hline & 0 & 0 & -2/5 & -1 & 1 & -64 \end{array}$$

- How many variables (original and slack) can be different from zero?

**Solution:**

at most 2

- $(x_3, x_4) = (0, 0)$  are non basic, what does this tell us about the constraints?

**Solution:**

They are active because their dual values are not zero

Let's generalize the previous case. Consider an LP with  $m$  constraints,  $n$  original variables and  $m$  slack variables. In an optimal solution:

- if  $m > n$ , how many variables (original and slack) can be nonzero at most?

**Solution:**

at most  $m$

- if  $m < n$  how many original variables must be zero at least? In other terms, in a mix planning problem with  $n$  products and  $m$ ,  $m < n$  resources, how many products at most will be to be produced in an optimal solution?

**Solution:**

$n - m$ , and hence at most  $m < n$  products

**Solution:**

at most  $m$



12. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{ll} \max & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & -z & b \\ \hline x_3 & 0 & 0 & 1 & 1/2 & 0 & 1 \\ x_1 & 1 & 1 & 0 & -1/2 & 0 & 1 \\ \hline & 0 & -2 & 0 & 1/2 & 1 & -1 \end{array}$$

$(x_2, x_4) = (0, 0)$  is non basic, what does this tell us about the constraints?

**Solution:**

The second constraint is active because its slack  $x_4$  is zero.  $x_2 = 0 \implies x_2 \geq 0$  is active.

13. If in the original space of the problem we had 3 variables, and there are 6 constraints, how many constraints would be active?

**Solution:**

3 constraints. With slack variables we would have 6 variables in all, if any of them is positive the constraint  $x_i \geq 0$  of the original variables would be active, otherwise the corresponding constraints of the original problem are active.

14. For the general case with  $n$  original variables:  
One basic feasible solution  $\iff$  a matrix of active constraints has rank  $n$ . True or False?

**Solution:**

True

15. Consider an LP problem with  $m$  constraints and  $n$  original variables,  $m > n$ . We saw that in  $\mathbb{R}^n$  a point is the intersection of at least  $n$  hyperplanes. In LP this corresponds to say that in a vertex there are  $n$  active constraints. Let a tableau be associated with a solution that makes exactly  $n + 1$  constraints active, what can we say about the corresponding basic and non-basic variable values?

**Solution:**

one basic variable is zero. Indeed, in the simplex we will have  $m + n$  variables and  $m$  variables in basis. We saw that the  $n$  non basic variables are set to zero and that there is an active constraint for each of them. Hence, if there are  $n + 1$  active constraints, there must be another variable that is set to zero. It must be a basic variable.

16. Given a polyhedron, what is the algebraic definition of vertex adjacency in 2, 3 and  $n$  dimensions?

**Solution:**

two vertices are adjacent iff:

- they have at least  $n - 1$  active constraints in common
- rank of common active constraints is  $n - 1$

How does this condition translate in terms of tableau?

**Solution:**

For what seen above this translates in  $n - 1$  variables in common in the tableau

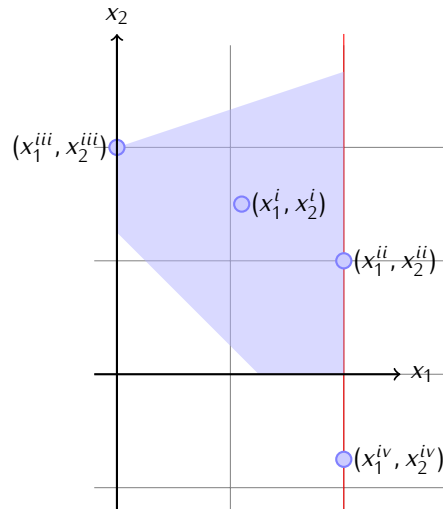
## Exercise 7

What argument is used to prove that the simplex algorithm always terminates in a finite number of iterations if it does not encounter a situation in which one of the basic variables is zero? What may happen instead if the latter situation arises and which remedies are introduced?

**Exercise 8\***

Exercise 3 from Exam 2013.  
Consider the following LP problem:

$$\begin{aligned}
 (P) \quad & \max z = x_1 + 5x_2 \\
 & \text{s.t.} \quad -x_1 + 3x_2 \leq 6 \\
 & \quad \quad 4x_1 + 4x_2 \geq 5 \\
 & \quad \quad 0 \leq x_1 \leq 2 \\
 & \quad \quad x_2 \geq 0
 \end{aligned}$$



(Note: the following subtasks can be carried out independently; use fractional mode for numerical calculations; in the online version you find the problem in ASCII format).

- a. The polyhedron representing the feasibility region is depicted in the figure. Indicate for each of the four points represented whether they are feasible and/or basic solutions. Justify your answer.

**Solution:**

- Point 1 is a feasible solution but not basic (no constraint is active in that point).
- Point 2 is a feasible solution but not basic (only one constraint is active while two are needed)
- Point 3 is a basic feasible solution
- Point 4 is a basic solution (combination of two active constraints) but non feasible.

- b. Write the initial tableau or dictionary for the simplex method. Write the corresponding basic solution and its value. State whether the solution is feasible or not and whether it is optimal or not.

**Solution:**

$$\begin{aligned}
 z = \max \quad & x_1 + 5x_2 \\
 \text{s.t.} \quad & -x_1 + 3x_2 \leq 6 \\
 & -4x_1 - 4x_2 \leq -5 \\
 & x_1 \leq 2 \\
 & x_1 \geq 0 \\
 & x_2 \geq 0
 \end{aligned}$$

	x1	x2	x3	x4	x5	-z	b
I	-1	3	1	0	0	0	6
II	-4	-4	0	1	0	0	-5
III	1	0	0	0	1	0	2
IV	1	5	0	0	0	1	0

The basic solution is  $x_1 = 0, x_2 = 0, x_3 = 6, x_4 = -5$  and  $x_5 = 2$ . Its value is 0. The solution is not feasible.

- c. Consider the following tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$-z$	$b$
I	0	4	1	$-1/4$	0	0	$29/4$
II	1	1	0	$-1/4$	0	0	$5/4$
III	0	$-1$	0	$1/4$	1	0	$3/4$
IV	0	4	0	$1/4$	0	1	$-5/4$

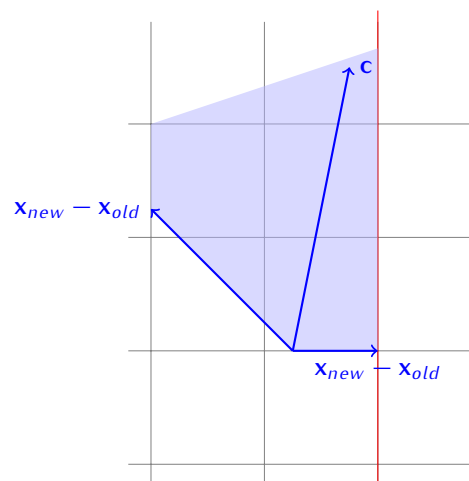
and the following three pivoting rules:

- largest coefficient
- largest increase
- steepest edge.

Which entering and leaving variables would each of them indicate? In this specific case, which rule would be convenient to follow? Report the details of the computations for the first two rules and carry out graphically the application of the third rule using the plot in the figure above (tikz code to reproduce the figure available in the online version.)

**Solution:**

- The two candidate entering variables are  $x_2$  and  $x_4$ . The reduced cost of  $x_2$  is larger hence that is the entering variable. The leaving variable is consequently given by the ratio test and is  $x_1$  since  $5/4 < 29/16$ .
- The two candidate entering variables are  $x_2$  and  $x_4$ . The increase possible with  $x_2$  is  $\min\{29/4 \cdot 1/4, 5/4 \cdot 1\} \cdot 4 = 5$  while the increase with  $x_4$  is  $\min\{3/4 \cdot 4\} \cdot 1/4 = 3/4$ . Hence  $x_2$  is the entering variable and the leaving variable is  $x_1$ .
- In the figure we plot the vector  $c$  which is the perpendicular to the objective function and the two vectors corresponding to the movement we would take by the iteration of the simplex. The angle between  $c$  and  $x_{new} - x_{old}$  is smaller for the decision  $x_2$  entering  $x_1$  leaving.



None of the three rules is convenient, the best would be to let  $x_4$  enter and  $x_5$  leave, we would reach the optimal solution in less iterations.

### Exercise 9\*

The two following LP problems lead to two particular cases when solved by the simplex algorithm. Identify these cases and characterize them, that is, give indication of which conditions generate them in general.

$$\begin{aligned} &\text{maximize} && 2x_1 + x_2 \\ &\text{subject to} && x_2 \leq 5 \\ &&& -x_1 + x_2 \leq 1 \\ &&& x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{maximize} && x_1 + x_2 \\ &\text{subject to} && 5x_1 + 10x_2 \leq 60 \\ &&& 4x_1 + 4x_2 \leq 40 \\ &&& x_1, x_2 \geq 0 \end{aligned}$$

**Solution:**

In the initial tableau of the first problem there is a column, the first one, with positive reduced cost and no positive  $a_{ij}$  term. This means that the corresponding variable  $x_1$  can be brought into the basis but the increase of its value is unlimited. This indicates that we have an unbounded problem.

The second LP problem is developed in the slides for the lecture on exception handling. After some iterations we reach a tableau in which a non basic variable has reduced cost zero. This indicates that it can be brought in the basis without a change in the objective function. Since the solution changes when we bring the variable in basis then the problem has more than one solution and it has therefore infinite solutions. They can be expressed as the convex combination of all optimal basic solutions.

**Exercise 10\***

Consider the following problem:

$$\begin{aligned} \max \quad & z = 4x_2 \\ \text{s.t.} \quad & 2x_2 \geq 0 \\ & -3x_1 + 4x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- a. Write the LP in equational standard form and say why it does not provide immediately an initial feasible basis for the simplex method.

**Solution:**

In the equational standard form we have a negative  $b$  term. The implication of this is that the initial solution of the simplex is infeasible because  $x_B \not\geq 0$ . If we try to make the term positive we end up not having an identity matrix in the tableau.

- b. To overcome the situation of infeasible basis construct the auxiliary problem for a phase I-phase II solution approach. Determine which variables are initially in basis and which are not in basis in the auxiliary problem.
- c. Answer the following questions
- i) Is the initial basis in the auxiliary problem feasible in the original problem?
  - ii) Is it optimal in the auxiliary problem?
  - iii) Is it degenerate?
  - iv) Can we say at this stage if phase I will terminate?
  - v) If it will terminate, can we say at this stage that it will terminate with a basis that corresponds to a feasible solution in the original problem?
  - vi) Solve the problem by carrying out Phase I and Phase II of the simplex algorithm.